Spring 2020, Math 621: Week 7 Problem Set
Due: Monday, March 23th, 2020
Ehrhart Polynomials and Ehrhart Series

Discussion problems. The problems below should be worked on in class.

(D1) *Ehrhart from Hilbert.* The goal of this problem is to prove Ehrhart's theorem.

(a) As is always **strongly** suggested, we begin with an example. Let $C \subset \mathbb{R}^3_{\geq 0}$ denote the cone over the simplex $P = \text{conv}\{(0,0),(2,1),(3,2),(0,3)\}$, and let $S = C \cap \mathbb{Z}^3$ denote the associated semigroup.

(i) Draw P, $2P$, and $3P$. List a few integer points in each corresponding slice of C.

(ii) Find all 9 minimal generators of S.

(iii) Applying Hilbert's theorem to $\Bbbk[S] \subset \Bbbk[x,y,z]$ under the fine grading, what form do we obtain for $\text{Hilb}(\Bbbk[S]; z_1, z_2, z_3)$ **without** computing a precise numerator?

(iv) What grading should we choose in the previous part to instead obtain $\text{Ehr}(P; z)$? Why does this not quite yield Ehrhart's theorem for P?

(v) Explain why $\Bbbk[x,y,z]$ is a module over $\Bbbk[S]$. Is it finitely generated?

(vi) Explain why $M = \Bbbk[S]$ is a module over $R = \Bbbk[z, x^2yz, x^3y^2z, y^3z]$, and find all 7 minimal homogeneous generators of M (as an R-module).

(vii) Apply Hilbert's theorem to the module M. What does this tell us about the rational form of the Ehrhart series of P?

(b) Using the ideas in the previous part, outline a proof for the following theorem.

Theorem. *Any rational cone $C \subset \mathbb{R}^d$ with extremal ray vectors $r_1, \ldots, r_k \in \mathbb{Z}^d_{\geq 0}$ has*

$$\sum_{p \in C \cap \mathbb{Z}^d} z^p = \frac{h(z)}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}$$

for some polynomial $h(z)$.

(c) Let $C \subset \mathbb{R}^3$ denote the cone over the unit square $P = [0,1]^2$. Apply the above theorem to C. Why does this not quite prove Ehrhart's theorem for P?

(d) Divide P into two triangles by drawing in one of the two diagonals. This is called a *triangulation* of P. Demonstrate that there exists a polynomial $h(z)$ such that

$$\text{Ehr}(P; z) = \frac{h(z)}{(1-z)^3}$$

by writing $\text{Ehr}(P; z)$ in terms of the Ehrhart series of 3 simplices.

(e) Briefly justify the following theorem (a classical result from polyhedral geometry) in the special case where P is an arbitrary rational polygon.

Theorem. *Any rational polytope P can be written as the union of rational simplices $T_1, \ldots, T_r$ such that (i) the vertices of each T_i coincide with vertices of P, (ii) each intersection $T_i \cap T_j$ is a face of both T_i and T_j, and (iii) $\text{vol}\, T_1 + \cdots + \text{vol}\, T_r = \text{vol}\, P$.*

(f) Putting everything together, prove that for any d-dimensional rational polytope P with denominator p, we have

$$\text{Ehr}(P; z) = \frac{h(z)}{(1 - z^p)^{d+1}}$$

for some polynomial $h(z)$. In particular, this proves Ehrhart's theorem.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove the following. You may use any theorems we have seen involving Ehrhart polynomials, including ones we have not (yet) proven.

Theorem (Pick's Theorem)**.** *For any lattice polygon P with I interior lattice points, B boundary lattice points, and area A, we have*

$$A = I + \tfrac{1}{2}B - 1.$$

(H2) Find, for each $h \in \mathbb{Z}_{\geq 1}$, the Ehrhart polynomial and Ehrhart series of

$$P = \mathrm{conv}\{(0,0,0),\ (0,0,1),\ (0,1,0),\ (h,1,1)\}$$

(this is known as *Reeve's tetrahedron*). You may use any theorems we have seen involving Ehrhart polynomials, including ones we have not (yet) proven.

(H3) Fix an affine semigroup $S \subset \mathbb{Z}^d_{\geq 0}$, and consider the semigroup algebra

$$R = \Bbbk[S] \subset \Bbbk[x_1, \ldots, x_d].$$

Characterize the monomial ideals $P \subset R$ that are prime.

(H4) Fix a rational polyhedron P, and for simplicity, assume $P \subset \mathbb{R}^d_{\geq 0}$. It is known that $P = Q + C$ for some rational polytope $Q \subset \mathbb{R}^d_{\geq 0}$ and some rational cone $C \subset \mathbb{R}^d_{\geq 0}$. Use this fact to prove that

$$\sum_{p \in P \cap \mathbb{Z}^d} z^p = \frac{h(z)}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}$$

for some polynomial $h(z)$ and some $r_1, \ldots, r_k \in \mathbb{Z}^d_{\geq 0}$.

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Pick's Theorem also holds for rational polygons.

(b) For any d-dimensional rational simplex $P \subset \mathbb{R}^d$, the Ehrhart series

$$\mathrm{Ehr}(P; z) = \frac{h(z)}{(1 - z)^{d+1}}$$

for some polynomial $h(z)$ with non-negative integer coefficients.