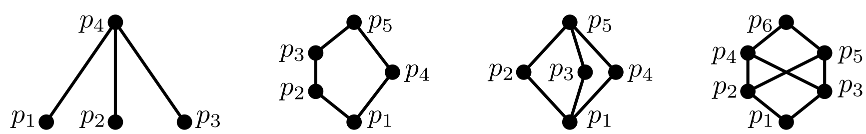
Spring 2020, Math 621: Week 8 Problem Set
Due: Friday, March 27th, 2020
Applications to Enumerative Combinatorics

Discussion problems. The problems below should be worked on in groups, but will not be submitted for credit. Only submit the homework problems at the end of this document. The content included covers enough material to complete the assigned problems, but if you are interested in further reading, I suggest Chapter 1 of *Combinatorial Reciprocity Theorems* by Matthias Beck and Raman Sanyal (freely available from the first author's webpage).

This week, we will be examining an application of polytopes and Ehrhart's theorem to combinatorial objects called posets. First, the obligatory (and less useful) technical definition: a *partially ordered set* (or *poset*) is a set Π with a partial order $\preceq$ that is reflexive, antisymmetric, and transitive (here, "partial" means that some elements $p, q \in \Pi$ are *incomparable*, in that neither $p \preceq q$ nor $q \preceq p$).

Having stated the technical definition, note that this is **not** the best way to think of a poset! For all of our posets, $\Pi = \{p_1, \ldots, p_d\}$ will be finite, and the easiest way to specify/think about a **finite** poset is with its *Hasse diagram*, wherein each point is an element of Π, and two elements p_i and p_j lie "above" each other whenever $p_i \preceq p_j$. Below are several examples of Hasse diagrams, and the posets they define will be used frequently through the rest of this sheet.



Π_1 Π_2 Π_3 Π_4

Some notes and useful terminology.

- We use $\preceq$ (as opposed to $\prec$) for a partial order since reflexivity is required (each $p_i \preceq p_i$). When we write $p_i \prec p_j$, we mean that $p_i \preceq p_j$ and $p_i \neq p_j$.
- If $p_i \prec p_j$ and there are no elements strictly between p_i and p_j, we say p_j *covers* p_i, and refer to this as a *cover relation*.
- In the Hasse diagram, we only draw edges for cover relations. For example, in the poset Π_2, we have $p_1 \preceq p_3$ since $p_1 \preceq p_2 \preceq p_3$ in the diagram, even though there is no edge directly between p_1 and p_3. (Which poset axiom ensures we can conclude this?)
- We say Π is *linear* if it happens to be a total ordering (i.e., no elements are incomparable). A *linear extension* of a poset Π is a linear poset $(\Pi', \preceq')$ obtained from Π by including additional relations. For example, $p_1 \preceq' p_2 \preceq' p_3 \preceq' p_4$ and $p_2 \preceq' p_1 \preceq' p_3 \preceq' p_4$ are both linear extensions of Π_1. (What does the Hasse diagram of a linear poset look like?)

We are now ready to bring polytopes into the mix. Given a poset $(\Pi, \preceq) = \{p_1, \ldots, p_d\}$, the *order polytope* of Π, denoted $\mathcal{O}(\Pi)$, is defined by

$$\mathcal{O}(\Pi) = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : 0 \leq a_i \leq 1 \text{ for } i = 1, \ldots, d, \text{ and } a_i \leq a_j \text{ whenever } p_i \preceq p_j\}.$$

In particular, the points in $\mathcal{O}(\Pi)$ have coordinates indexed by the elements of Π, and the relations in Π determine the hyperplanes bounding $\mathcal{O}(\Pi)$.

(D1) *H-descriptions of order polytopes.* The goal of this problem is to "prune" the inequalities in the definition of $\mathcal{O}(\Pi)$ above, keeping only the ones that actually bound facets, by eliminating any inequalities that are implied by others.

(a) Write down all 13 relations in the poset Π_2 above.

(b) Write down all 11 inequalities in the definition of $\mathcal{O}(\Pi)$ for the poset Π_1 above.

(c) Determine the H-description of $\mathcal{O}(\Pi)$, that is, the *irredundant* list of facet inequalities. Your answer will depend on the Hasse diagram of Π.

Hint: the posets Π_1, Π_2, Π_3, and Π_4 have 7, 7, 8, and 10 facets, respectively.

(D2) *Faces of order polytopes.* Fix a poset $(\Pi, \preceq) = \{p_1, \ldots, p_d\}$.

(a) Determine for which Π the order polytope $\mathcal{O}(\Pi)$ is a simplex (that is, when $\mathcal{O}(\Pi)$ has exactly $d+1$ facets, or (equivalently) exactly $d+1$ vertices).

(b) Prove that $\dim \mathcal{O}(\Pi) = d$. Recall that do to this, we must find d linearly independent vectors of the form $x - y$ with $x, y \in \mathcal{O}(\Pi)$.

Given a poset $(\Pi, \preceq)$, a function $f : \Pi \to \{1, \ldots, n\}$ is *(weakly) order-preserving* if

$$p \preceq q \qquad \text{implies} \qquad f(p) \le f(q) \qquad \text{for all} \qquad p, q \in \Pi.$$

Let $\Omega_\Pi(n)$ denote the number of order preserving functions $\Pi \to \{1, \ldots, n\}$. For example, the function $f : \Pi_2 \to \{1, \ldots, 8\}$ given by

$$f(p_1) = 1, \quad f(p_2) = 4, \quad f(p_3) = 6, \quad f(p_4) = 3, \quad \text{and} \qquad f(p_5) = 6$$

is order preserving, but if we change $f(p_3) = 2$, the resulting function would not be order preserving since $p_2 \preceq p_3$ but $f(p_2) > f(p_3)$.

(D3) *Ehrhart functions of order polytopes.* The goal of this problem is to prove that for any finite poset $(\Pi, \preceq) = \{p_1, \ldots, p_d\}$, the function $\Omega_\Pi(n)$ is a polynomial in n of degree d.

(a) Find $\Omega_\Pi(2)$ for the posets Π_1 and Π_2 by listing all order preserving functions to $\{1, 2\}$.

(b) Find all points in $\mathcal{O}(\Pi) \cap \{0, 1\}^d$ for $\Pi = \Pi_1$ and $\Pi = \Pi_2$. What do you notice?

(c) Find a relationship between $L_{\mathcal{O}(\Pi)}(n)$ and $\Omega_\Pi(n)$, and prove your claim by writing down an explcit bijection. What does Ehrhart's theorem then tell you about $\Omega_\Pi(n)$?

(d) Give a combinatorial interpretation of $L^\circ_{\mathcal{O}(\Pi)}(n)$ (that is, characterize which order preserving functions correspond to interior points under your bijection). Apply Ehrhart reciprocity to obtain another theorem.

(e) What surprising relationship does this imply between the number of weakly order preserving functions on Π and the number of strictly order preserving functions on Π?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Determine a complete list of 3-element posets (there are 5 distinct Hasse diagrams, and only one poset for each is required).

(H2) Fix a poset $(\Pi, \preceq) = \{p_1, \ldots, p_d\}$.

(a) Prove the vertices of $\mathcal{O}(\Pi)$ lie in $\{0,1\}^d$.
Hint: for each point $x \in \mathcal{O}(\Pi) \setminus \{0,1\}^d$, locate a tiny line segment L with $x \in L \subset \mathcal{O}(\Pi)$. Why does this imply x cannot be a vertex?

(b) As a consequence of the previous part, the vertices of $\mathcal{O}(\Pi)$ are precisely the 01-vectors that satisfy the inequalties defining $\mathcal{O}(\Pi)$. One interpretation of this is we can associate to each vertex v of $\mathcal{O}(\Pi)$ a particular subset of Π (for example, a vertex $v = (1,0,0,1,0,1)$ corresponds to the set $\{p_1, p_4, p_6\} \subset \Pi$). Characterize which subsets of Π coorespond to vertices of $\mathcal{O}(\Pi)$ in terms of the Hasse diagram of Π.

(H3) Recall that a *triangulation* of a polytope P is an expression of P as a union of simplices whose pairwise intersections are faces. Prove that any order polytope has a triangulation consisting of order polytopes.

(H4) Read, digest, and transcribe the direct (i.e., polytope-free) proof in Proposition 1.3.1 of *Combinatorial Reciprocity Theorems* by Beck and Sanyal (this requires filling in some gaps).

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Given a poset Π, characterize the edges of the order polytope $\mathcal{O}(\Pi)$.