Spring 2020, Math 621: Week 10 Problem Set
Due: Friday, April 17th, 2020
Gröbner Bases (Deep Cuts)

Warmup and Discussion problems. The problems below should be worked on in groups, but will not be submitted for credit. Only submit the homework problems at the end of this document. Try to read up through the start of the first discussion problem, and complete the warmup problems, prior to starting with your group. The content included covers enough material to complete the assigned problems, but if you are interested in further reading, I suggest Chapter 2 of *Ideals, Varieties, and Algorithms* by David Cox, John Little, and Donal O'Shea, Chapters 1 and 5 of *Using Algebraic Geometry* by David Cox, John Little, and Donal O'Shea, or Chapter 15 of *Commutative Algebra with a View Toward Algebraic Geometry* by David Eisenbud.

(W1) Use Buchberger's algorithm to obtain a Gröbner basis for

$$I = \langle x^2 - y^2, x^4y - xy^3 \rangle \subset \Bbbk[x, y]$$

under the glex term order. Your answer should consist of 3 polynomials.

Now that we know how to obtain Gröbner bases for a given ideal, we turn our attention to questions of minimality. Indeed, the output of Buchberger's algorithm can often be pruned, sometimes substantially. A Gröbner basis G with respect to $\prec$ is called *reduced* if (i) for all $g_i, g_j \in G$, no term of g_i is divisible by the leading term of g_j, and (ii) the coefficient of each leading term is 1.

(D1) *Reduced Gröbner bases.*

(a) Argue that if $g_1, \ldots, g_r$ is a Gröbner basis for an ideal I, and g_r' is the remainder after dividing g_r by $g_1, \ldots, g_{r-1}$, then $g_1, \ldots, g_{r-1}, g_r'$ is also a Gröbner basis for I.

(b) Use the previous part to find a reduced Gröbner basis for the ideal

$$J = \langle x^3 - y^2, x^4y^4 - z^3, xy^6 - z^3, y^8 - x^2z^3 \rangle \subset \Bbbk[x, y, z]$$

under glex order. You may assume the given generating set is a Gröbner basis (it is).

(c) Can you give a general algorithm for finding a reduced Gröbner basis for a given ideal?

(d) The *initial ideal* of an ideal I under $\prec$ is the monomial ideal

$$\mathrm{In}_\prec(I) = \langle \mathrm{In}_\prec(f) : f \in I \rangle$$

generated by the initial terms of **every** element of I. Argue that $G = \{g_1, \ldots, g_r\}$ is a Gröbner basis for I under $\prec$ if and only if we have $\mathrm{In}_\prec(I) = \langle \mathrm{In}_\prec(g_1), \ldots, \mathrm{In}_\prec(g_r) \rangle$.

(e) Using the previous part, find a (the) minimal generating set of $\mathrm{In}_\prec(J)$.

(f) Fix an ideal $I \subset \Bbbk[x_1, \ldots, x_k]$ and a term order $\prec$. The goal of this part is to prove that there is a **unique** reduced Gröbner basis for I under $\prec$.

(i) Prove that the leading terms of the polynomials in a reduced Gröbner basis for I under $\prec$ are precisely the minimal generators of the monomial ideal $\mathrm{In}_\prec(I)$.

(ii) Suppose G and G' are two reduced Gröbner bases for I, and that $f \in G$ and $g \in G'$ have the same leading term. Argue that $f - g = 0$.

(iii) Conclude that there is a unique reduced Gröbner basis for I under $\prec$.

(g) Prove that if I is a binomial ideal (i.e., I can be generated by differences of monomials) and $\prec$ is any term order, then the reduced Gröbner basis of I with respect to $\prec$ is comprised entirely of binomials.

One interesting consequence of the last part above an algorithm to determine whether a given ideal I is binomial: simply pick any term order $\prec$, compute the reduced Gröbner basis G for I with respect to $\prec$, and check if G consists entirely of binomials. Even if, say, our starting list of generators for I contains some non-binomials, as long as it is possible to find a generating set consisting of binomials for I, then the reduced Gröbner basis will consist of binomials.

(D2) *Gröbner bases of modules.* Let $R = \Bbbk[x_1, \ldots, x_k]$, and let $e_1, \ldots, e_n$ are the standard basis vectors of the free module R^n. The goal of this problem is to extend the concept of a Gröbner basis from ideals in R to submodules of R^n. This will play a key role in the proof of the Hilbert Syzygy Theorem in a few short weeks.

(a) Recall that a *monomial* in R^n is an element of the form $x^a e_i$ (that is, a monomial in R times a standard basis vector of R^n). Decide what it means for $x^a e_i$ to divide $x^b e_j$.

(b) Given a term order $\prec$ on R, define the *position-over-term* (POT) order $\prec_{\text{pot}}$ on R^n such that $x^a e_i \prec_{\text{pot}} x^b e_j$ whenever (i) $i > j$ or (ii) $i = j$ and $x^a \prec x^b$. Determine the initial terms of

$$m = x^4 y e_1 + x^3 y^3 e_1 + y^2 e_1 + x^4 y e_2 + x y^3 e_2 \qquad \text{and} \qquad m' = xye_1 - x^3 e_2$$

under pot-glex order, and use the division algorithm to divide m by m'.

(c) Given a term order $\prec$ on R, define the *term-over-position* (TOP) order $\prec_{\text{top}}$ on R^n such that $x^a e_i \prec_{\text{top}} x^b e_j$ whenever (i) $x^a \prec x^b$, or (ii) $x^a = x^b$ and $i > j$. Determine the initial terms of m and m' from the previous part under top-glex order, and use the division algorithm to divide m by m'.

(d) Using the above as intuition, decide on a reasonable definition of a *term order* $\prec$ on R^n.

(e) Give a reasonable definition of a *Gröbner basis* $m_1, \ldots, m_r$ of a submodule $M \subset R^n$ with respect to a given term order $\prec$ on R^n.

(f) Identify a reasonable element in R^2 to serve as the sygyzgy

$$S(x^2 y e_1 + x^5 e_2, y^3 e_1 + x^2 e_2 - y e_2)$$

under the pot-glex order. Is there a reasonable choice for $S(x^2 e_1 + y e_2, y^2 e_2 + x e_2)$? Use your intuition to carefully define the *syzygy* element $S(m, m')$, keep in mind that we want the following theorem to hold.

Theorem. *A list of elements $m_1, \ldots, m_r$ is a Gröbner basis for $M \subset R^n$ if and only if division of each syzygy $S(m_i, m_j)$ by $m_1, \ldots, m_r$ yields remainder 0.*

(g) Compute a reduced Gröbner basis for the submodule

$$M = \left\langle x^2 e_1 - y^2 e_1,\, xye_1 - ye_2 \right\rangle \subset R^2$$

under the pot-glex term order.

(h) Decide on a definition of the *initial submodule* $\mathrm{In}_\prec(M)$. Find this in the above example.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Determine whether the ideal

$$I = \langle x^2 - y^2, x^3y^4 - xy^6 + x^3y, x^4y^3 - x^2y^5 + xy^3 \rangle \subset \Bbbk[x, y, z]$$

can be generated by binomials.

(H2) Fix an ideal $I \subset \Bbbk[x_1, \ldots, x_k]$, let $\prec$ denote the lex term order with $x_1 \prec x_2 \prec \cdots \prec x_k$, and let G denote the reduced Gröbner basis for I. Prove that the reduced Gröbner basis for $I' = I \cap \Bbbk[x_2, \ldots, x_k]$ with respect to $\prec$ is $G' = G \cap \Bbbk[x_2, \ldots, x_k]$. Does the same hold if $\prec$ is the glex term order?

(H3) Suppose $I \subset \Bbbk[x_1, \ldots, x_k]$ is generated by polynomials of the form $a_1x_1 + \cdots a_kx_k$ with $a_1, \ldots, a_k \in \Bbbk$. Find a formula for the number of elements in the reduced Gröbner basis for I under any given term order $\prec$.

(H4) Prove that for any homogenous ideal $I \subset \Bbbk[x_1, \ldots, x_k]$ under the standard grading and any term order $\prec$, we have

$$\mathrm{Hilb}(R/I; z) = \mathrm{Hilb}(R/\operatorname{In}_\prec(I); z).$$

Does the same necessarily hold if we use a grading other than the standard grading?

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If R is graded and I is homogenous, then the reduced Gröbner basis of I under any term order $\prec$ is comprised entirely of homogeneous elements.

(b) For $R = \Bbbk[x, y]$, the given generating set for the submodule

$$\big\langle (xy + 4x)e_1 + x^2e_3,\ (y - 1)e_2 + (x - 2)e_3 \big\rangle \subset R^3$$

is a Gröbner basis under both the pot-lex term order and the top-lex term order.