Spring 2020, Math 621: Week 11 Problem Set
Due: Friday, April 24th, 2020
Noetherian Rings and the Hilbert Basis Theorem

Warmup and Discussion problems. The problems below should be worked on in groups, but will not be submitted for credit. Only submit the homework problems at the end of this document. Try to read up through the start of the first discussion problem, and complete the warmup problems, prior to starting with your group. The content included covers enough material to complete the assigned problems, but if you are interested in further reading, I suggest Chapter 2 of *Ideals, Varieties, and Algorithms* by David Cox, John Little, and Donal O'Shea, or Chapter 9 and 12 of *Abstract Algebra* by David Dummit and Richard Foote.

This week, we will prove the first of the two big theorems of Hilbert, called the Hilbert Basis Theorem, which states that every ideal $I \subset \Bbbk[x_1, \ldots, x_k]$ is finitely generated. This, together with the Hilbert Syzygy Theorem (coming soon), implies the "Big Hilbert's Theorem" from week 4. Before we see the proof, we will examine the large family of rings with this property.

A commutative ring R is said to be *Noetherian* if every ideal $I \subset R$ is finitely generated. There are several other equivalent definitions of Noetherian, and our first task this week will be to prove their equivalence.

Theorem. *For any (commutative) ring R, the following are equivalent:*

(a) R is Noetherian (that is, every ideal in R is finitely generated);

(b) R satisfies the ascending chain condition (ACC) on ideals (i.e., any chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes, meaning for some N we have $I_k = I_{k+1}$ for all $k \geq N$); and

(c) every nonempty set of ideals of R has a maximal element under containment.

The classic example of a non-Noetherian ring is $\Bbbk[x_1, x_2, \ldots]$, the polynomial ring in infinitely many variables. This ring violates all 3 parts of the above theorem: the ideal $\langle x_1, x_2, x_3, \ldots\rangle$ is not finitely generated, the ascending chain

$$\langle x_1\rangle \subsetneq \langle x_1, x_2\rangle \subsetneq \langle x_1, x_2, x_3\rangle \subsetneq \cdots$$

does not eventually stabilize, and the set $\{I_k : k \geq 0\}$ of ideals $I_k = \langle x_1, x_2, \ldots, x_k\rangle$ has no maximal element under containment. This single example actually encapsulates most of the ideas in the proof of the above theorem, as we will see in Problem (D1).

(W1) Give a brief argument that $\mathbb{Z}$ satisfies the ascending chain condition on ideals.

(D1) *Equivalent definitions.* In this problem, we will work through a proof of the above theorem.

(a) Play a few rounds of the "ascending chain game": starting with $I_1 = \langle x^2y^3\rangle \subset \Bbbk[x, y]$, take turns selecting the next ideal in the chain by adding one additional generator (which may or may not override existing generators). Verify that you eventually get "trapped" and become unable to add further elements (i.e., that the ascending chain condition holds). For simplicity, you may restrict your attention to monomial ideals.

(b) Read the following proof that if R has an ideal I that is not finitely generated, then there exists an ascending chain of ideals that does not stabilize. Prove the converse.

Proof. Fix $f_1 \in I$. Since I is not finitely generated, $I \supsetneq \langle f_1 \rangle$, so there exists some $f_2 \in I \setminus \langle f_1 \rangle$. Likewise, we must have $I \supsetneq \langle f_1, f_2 \rangle$, so there exists some $f_3 \in I \setminus \langle f_1, f_2 \rangle$. Continuing in this way, we obtain an ascending chain

$$\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots$$

that does not stabilize since each ideal contains at least one new element f_i. □

(c) Prove that if R has some ascending chain of ideals that does not eventually stabilize, then R also has a set of ideals that does not contain a maximal element.

(d) Complete the proof below that if R has a set of ideals that does not contain a maximal element, then R also has an ascending chain of ideals that does not eventually stabilize.

Proof. Fix a nonempty set S of ideals, and fix $I_1 \in S$. By assumption, I_1 is not maximal under containment among the ideals in S, so there exists $I_2 \in S$ with $I_1 \subsetneq I_2$.
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(e) Conclude that the theorem holds.

(f) Use the above theorem to prove that if R is Noetherian, then for any infinite list $f_1, f_2, \ldots \in R$ of elements, we have $\langle f_1, f_2, \ldots \rangle = \langle f_1, \ldots, f_N \rangle$ for some N (in particular, any infinite generating set for an ideal I has a finite subset that also generates I).
Note: this is the underlying reason Buchberger's algorithm terminates.

Noetherian rings can also be characterized in terms of their finitely generated modules. All parts of the theorem below bear a striking resemblence to the first theorem, and indeed most of the proofs are near identical. The primary hurdle is proving (a) implies (b), as we must jump from a condition on ideals to a condition on modules.

Theorem. *For any ring R, the following are equivalent.*

(a) R is Noetherian (that is, all 3 parts of the above theorem hold);

(b) for every finitely generated R-module M, every submodule $M' \subset M$ is also finitely generated;

(c) for every finitely generated R-module M, every ascending chain

$$M'_1 \subseteq M'_2 \subseteq M'_3 \subseteq \cdots$$

of submodules of M eventually stabilizes; and

(d) for every finitely generated R-module M, every nonempty set of submodules of M has a maximal element under containment.

Note that the assumption throughout that M is finitely generated is essential, regardless of whether R is Noetherian. Indeed, the module $M = \bigoplus_{i=1}^{\infty} R$ is not finitely generated, as are numerous submodules therein.

(D2) *Modules over Noetherian rings.*

(a) Demonstrate that $\Bbbk[x_1, x_2, \ldots]$ violates all parts of the above theorem.

(b) Adapt the proof form Problem (D1)(b) to prove that (b) implies (c) above.

(c) Adapt the proof form Problem (D1)(e) to prove that (d) implies (c) above.

(d) Give a 1-line proof that (b) implies (a) above.

(e) All that remains is to show (a) implies (b). Suppose every ideal in R is finitely generated, and let $M = \langle m_1, \ldots, m_k \rangle$ denote a finitely generated R-module.

(i) Locate a "natural" homomorphism $\varphi : R^k \to M$ from the free module R^k to M, and argue that any submodule $M' \subset M$ is finitely generated if and only if its preimage $\varphi^{-1}(M') \subset R^k$ is finitely generated. Conclude that it suffices to assume $M = R^k$ is free and each $m_i = e_i$.

(ii) Fix a submodule $M' \subset R^k$, and let

$$I = \{a : (a, a_2, \ldots, a_k) \in M' \text{ for some } a_2, \ldots, a_k \in R\} \subset R.$$

Prove that I is an ideal of R.

(iii) Prove that

$$M'' = \{(0, a_2, \ldots, a_k) \in M' : a_2, \ldots, a_k \in R\} \subset R^k$$

is a submodule of R^k.

(iv) Argue that M'' is isomorphic to a submodule of the free module R^{k-1}, and that by induction on k, we can assume M'' is finitely generated.

(v) Write $I = \langle b_1, \ldots, b_r \rangle$ for some $b_1, \ldots, b_r \in R$ (why can we do this?), and fix a list of elements $m'_1, \ldots, m'_r \in M'$ with first coordinates $b_1, \ldots, b_r$, respectively. Additionally, write $M'' = \langle m''_1, \ldots, m''_t \rangle$ for some elements $m''_1, \ldots, m''_t \in M$. Prove that $M' = \langle m'_1, \ldots, m'_r, m''_1, \ldots, m''_t \rangle$.

(D3) *The Hilbert Basis Theorem.* We are now ready to prove the Hilbert Basis Theorem. We will first prove the following stronger result (which is sometimes given the same name).

Theorem. *If R is Noetherian, then $R[x]$ is Noetherian.*

(a) Fix an ideal $I \subset R[x]$. Let $L \subset R$ denote the set of leading coefficients of elements of I. Argue that L is an ideal (convention: the 0 polynomial has leading coefficient 0).

(b) Let $L_d \subset R$ denote the set of leading coefficients of elements of I with degree at most d. Generalize your argument from the previous part to prove that L_d is an ideal.

(c) Argue that for some N, we have $L_d = L$ for all $d \geq N$.

(d) Let $a_1, \ldots, a_r$ denote a (finite) generating set for L, and fix polynomials $f_1, \ldots, f_r \in I$, each of degree at most N, with leading coefficients $a_1, \ldots, a_r$, respectively. Prove that $\langle f_1, \ldots, f_r \rangle$ contains every element of I of degree at least N.

Hint: suppose $f \in I \setminus \langle f_1, \ldots, f_r \rangle$ has minimal degree, and obtain a contradiction.

(e) Fix a degree $d \leq N$. Locate a list of polynomials $g_1, \ldots, g_t \in I$ such that $\langle g_1, \ldots, g_t \rangle$ contains every element of I of degree d.

(f) Conclude that I has a finite generating set.

(g) Give a brief inductive proof of the Hilbert Basis Theorem.

Theorem (Hilbert Basis Theorem)**.** *Every ideal $I \subset \Bbbk[x_1, \ldots, x_k]$ is finitely generated.*

We close with a brief historical tangent. Noetherian rings are named after Emmy Noether for her foundational contributions to commutative algebra. Prior to her work, numerous results on *primary decomposition* (like prime factorization, but for ideals) were obtained for small classes of rings, and often involved long, tedious, and specialized arguments. Emmy Noether identified the ascending chain condition as the core feature, and in doing so (i) simlutaneously extending their results to a single, much larger class of rings (i.e., Noetherian rings), and (ii) provided substantially shorter and cleaner arguments.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) An ideal $I \subset R$ is *irreducible* if $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(a) Prove that any prime ideal is irreducible.

(b) Prove that if R is Noetherian, then every ideal $I \subset R$ can be written as an intersection of finitely many irreducible ideals (such an expression is called an *irreducible decomposition* of I).

(H2) Let $R = \Bbbk[x_1, \ldots, x_k]$, and let $I = \langle x^{a_1}, \ldots, x^{a_r} \rangle \subset R$ denote a monomial ideal.

(a) Characterize when I is irreducible in terms of the exponent vectors $a_1, \ldots, a_r$.

(b) Find an irreducible decomposition of

$$I = \langle x^4, y^4, z^4, xy^2z^3, x^3yz^2, x^2y^3z \rangle \subset \Bbbk[x, y, z].$$

(H3) Finish any remaining parts of the proof of the Hilbert Basis Theorem from Problem (D3) that your group did not complete in class. Once you have completed this, it suffices to write "DONE" as your answer.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If R is a Noetherian ring and I is an ideal, then R/I is Noetherian.

(b) If R is a ring, $I \subset R$ is an ideal, and R/I is Noetherian, then R is Noetherian.

(c) Every ideal can be written as an intersection of finitely many prime ideals.

(d) Any Noetherian ring R also satisfies the descending chain condition (DCC) on ideals (that is, any descending chain

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

of ideals eventually stabilizes).