Spring 2020, Math 621: Week 12 Problem Set
Due: Monday, May 4th, 2020
Free Resultions and the Hilbert Syzygy Theorem

Warmup and Discussion problems. The problems below should be worked on in groups, but will not be submitted for credit. Only submit the homework problems at the end of this document. Try to read up through the start of the first discussion problem, and complete the warmup problems, prior to starting with your group. The content included covers enough material to complete the assigned problems, but if you are interested in further reading, I suggest Chapter 6 of *Using Algebraic Geometry* by David Cox, John Little, and Donal O'Shea, or Chapter 1 of *Combinatorial Commutative Algebra* by Ezra Miller and Bernd Sturmfels.

To date, we have used the words "generators" and "relations" quite a bit in passing while working through the material so far. We have a pretty good understanding of the former (and its precise meaning in different contexts); we have also developed a philosophical idea/intuition for the latter, though words like "minimal relation" we have avoided defining rigorously. This week, we begin our study of free resolutions, which are the formal mathematical objects that encapsulate relations-related information about the generators of a given module.

We begin with some formal definitions. A sequence of R-modules and homomorphisms

$$\cdots \xrightarrow{\varphi_{i-2}} M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} \cdots$$

is said to be *exact* if for each i, we have $\ker \varphi_i = \operatorname{Im} \varphi_{i-1}$ (i.e., the kernel of the map leaving M_i equals the image of the map coming into M_i). A *free resolution* of an R-module M is an exact sequence of the form

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots$$

wherein each module F_i is free. If there are only finitely many nonzero F_i in the sequence, as in

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_\ell \longleftarrow 0,$$

we say the *length* of the resolution is ℓ.

This "abstract nonsense" is best illustrated with some concrete examples. Let's first consider

$$I = \langle x^4y, x^2y^2, xy^3 \rangle \subset R = \Bbbk[x, y].$$

A free resolution for I can be constructed in one homological degree at a time. Starting with $0 \leftarrow I$, whose kernel is all of I, we choose $F_0 = R^3$ and send each basis vector of F_0 to a generator of I under φ_0, ensuring $\operatorname{Im} \varphi_0 = I$. This yields

$$0 \longleftarrow I \xleftarrow{\begin{array}{r} x^4y \longleftarrow\!\shortmid \mathbf{e}_1 \\ x^2y^2 \longleftarrow\!\shortmid \mathbf{e}_2 \\ xy^3 \longleftarrow\!\shortmid \mathbf{e}_3 \end{array}} R^3$$

so far. From there, we see $\ker \varphi_0$ must be generated by differences of monomials, since φ_0 sends monomials to monomials. The only minimal such elements are $y\mathbf{e}_1 - x^2\mathbf{e}_2$ and $y\mathbf{e}_2 - x\mathbf{e}_3$, meaning

$$\ker \varphi_0 = \langle y\mathbf{e}_1 - x^2\mathbf{e}_2, y\mathbf{e}_2 - x\mathbf{e}_3 \rangle.$$

As such, we choose $F_1 = R^2$ and define φ_1 to send $\mathbf{e}_1$ and $\mathbf{e}_2$ to these elements of F_0, obtaining

$$0 \longleftarrow I \xleftarrow{\begin{array}{r} x^4y \longleftarrow\!\shortmid \mathbf{e}_1 \\ x^2y^2 \longleftarrow\!\shortmid \mathbf{e}_2 \\ xy^3 \longleftarrow\!\shortmid \mathbf{e}_3 \end{array}} R^3 \xleftarrow{\begin{array}{r} (y\mathbf{e}_1 - x^2\mathbf{e}_2) \longleftarrow\!\shortmid \mathbf{e}_1 \\ (y\mathbf{e}_2 - x\mathbf{e}_3) \longleftarrow\!\shortmid \mathbf{e}_2 \end{array}} R^2.$$

Lastly, we see that $\ker \varphi_1 = 0$, so we are ready to complete the resolution.

$$0 \longleftarrow I \xleftarrow{\begin{array}{r} x^4y \longleftarrow\!\shortmid\ \mathbf{e}_1 \\ x^2y^2 \longleftarrow\!\shortmid\ \mathbf{e}_2 \\ xy^3 \longleftarrow\!\shortmid\ \mathbf{e}_3 \end{array}} R^3 \xleftarrow{\begin{array}{r} (y\mathbf{e}_1 - x^2\mathbf{e}_2) \longleftarrow\!\shortmid\ \mathbf{e}_1 \\ (y\mathbf{e}_2 - x\mathbf{e}_3) \longleftarrow\!\shortmid\ \mathbf{e}_2 \end{array}} R^2 \longleftarrow 0$$

The maps in this series are more concise to write as matrices, as in

$$0 \longleftarrow I \xleftarrow{\begin{bmatrix} x^4y & x^2y^2 & xy^3 \end{bmatrix}} R^3 \xleftarrow{\begin{bmatrix} y & 0 \\ -x^2 & y \\ 0 & -x \end{bmatrix}} R^2 \longleftarrow 0.$$

This illustrates an advantage of writing the maps in a free resolution right-to-left: if we write the elements of the free modules as column vectors, then we can use standard matrix multiplication. The same goes for composition of sequential maps (which should always be 0, due to exactness).

It is important to note that this is not the only free resolution for I. Indeed, suppose we had observed that $y^2\mathbf{e}_1 - x^3\mathbf{e}_3 \in \ker \varphi_0$, without noticing that

$$y^2\mathbf{e}_1 - x^3\mathbf{e}_3 = y(y\mathbf{e}_1 - x^2\mathbf{e}_2) + x^2(y\mathbf{e}_2 - x\mathbf{e}_3)$$

made it redundant as a generator for $\ker \varphi_0$, and chosen $F_1 = R^3$ with $\varphi_1(\mathrm{e}_3) = y^2\mathbf{e}_1 - x^3\mathbf{e}_3$. This would instead yield the partial resolution

$$0 \longleftarrow I \xleftarrow{\begin{bmatrix} x^4y & x^2y^2 & xy^3 \end{bmatrix}} R^3 \xleftarrow{\begin{bmatrix} y & 0 & y^2 \\ -x^2 & y & 0 \\ 0 & -x & -x^3 \end{bmatrix}} R^3$$

and now $\ker \varphi_1 \neq 0$ since $\varphi_1(\mathbf{e}_3) = \varphi_1(y\mathbf{e}_1 + x^2\mathbf{e}_2)$. As such, the free resolution

$$0 \longleftarrow I \xleftarrow{\begin{bmatrix} x^4y & x^2y^2 & xy^3 \end{bmatrix}} R^3 \xleftarrow{\begin{bmatrix} y & 0 & y^2 \\ -x^2 & y & 0 \\ 0 & -x & -x^3 \end{bmatrix}} R^3 \xleftarrow{\begin{bmatrix} y \\ x^2 \\ -1 \end{bmatrix}} R \longleftarrow 0.$$

is obtained. Though both free resolutions are perfectly valid, the second resolution highlights another important fact: some free resolutions are "smaller" than others. We will make this notion precise in Problem (D2) this week.

As a second example, consider

$$J = \langle x^3 - y^2, x^4y^4 - z^3 \rangle \subset R = \Bbbk[x, y, z].$$

We obtain the free resolution

$$0 \longleftarrow R/J \longleftarrow R \xleftarrow{\begin{bmatrix} x^3 - y^2 & x^4y^4 - z^3 \end{bmatrix}} R^2 \xleftarrow{\begin{bmatrix} x^4y^4 - z^3 \\ y^2 - x^3 \end{bmatrix}} R \longleftarrow 0.$$

for R/J. Unlike the first example, some matrix entries are not monomials. We will see that this stems from the grading on the module being resolved (indeed, I is a monomial ideal and thus finely graded, and while the entries of the matrices in the resolution for J are not monomial, they are homogeneous under the McNugget grading on J).

(W1) Use matrix multiplication to verify that the image of each map in the free resolutions above is contained in the kernel of the next map.

(D1) *Existence of free resolutions.* Throughout this problem, unless otherwise stated, R is any ring, and modules over R need not be finitely generated.

(a) Argue that the sequence $0 \to M \xrightarrow{\varphi} N$ is exact if and only if φ is injective. State and prove analogous results concerning (i) an exact sequence $M \xrightarrow{\varphi} N \to 0$, and (ii) an exact sequence $0 \to M \xrightarrow{\varphi} N \to 0$.

(b) A *short exact sequence* is an exact sequence of the form

$$0 \to K \to M \xrightarrow{\varphi} N \to 0.$$

Prove that $K \cong \ker\varphi$ and $N \cong M/\ker\varphi$ in any such sequence.

(c) Given below is a proof that for any ring R, any R-module M has a free resolution. Prove that if R is Noetherian and M is a finitely generated R-module, then we can pick the free modules F_i to have finite rank.

Proof. Pick any generating set $G \subset M$ for M (at worst, we could choose $G = M$). Let $F_0 = \bigoplus_{g\in G} R$, and begin the free resolution for M with $0 \leftarrow M \xleftarrow{\varphi_0} F_0$ using the surjective homomorphism $\varphi_0 : F_0 \to M$ defined by sending $\mathbf{e}_g \mapsto g$ for each $g \in G$. Inductively, suppose

$$0 \leftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} \cdots \xleftarrow{\varphi_j} F_j$$

is exact. As before, choose a generating set G for $\ker\varphi_j \subset F_j$, let $F_{j+1} = \bigoplus_{g\in G} R$, and define $\varphi_{j+1} : F_{j+1} \to F_j$ by $\mathbf{e}_g \mapsto g$ for each $g \in G$. This yields a free resolution

$$0 \leftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots$$

as desired. $\square$

(d) Suppose $R = \Bbbk[x,y]/\langle xy\rangle$ and $M = R/\langle x\rangle$. Find a free resolution for M.

(e) Suppose $R = \Bbbk[x,y,z]$, $I = \langle x,y,z\rangle$, and $M = R/I$. Find a free resolution for M.

(D2) *Graded free resolutions and minimality.* Suppose $R = \Bbbk[x_1,\ldots,x_k]$ is $\mathbb{Z}^d_{\geq 0}$-graded, and fix graded R-modules M and N. We say a map $\varphi : M \to N$ is *graded* if $\deg(f) = \deg(\varphi(f))$ for every homogeneous $f \in M$.

Throughout this problem, the ideals I and J refer to those defined before Problem (D1).

(a) Prove if φ is graded, then $\ker\varphi$ is a homogeneous submodule of M.

(b) Retrace through the proof in Problem (D1)(c). Conclude that every graded module M over a graded ring R has a free resolution in which each F_i and each φ_i are graded (we say such a free resolution is *graded*).

(c) The ideals in the examples prior to Problem (D1) are homogeneous, but technically the free resolutions we constructed are **not** graded. Why?

(d) Let's resolve this issue (pun intended). Find a free resolution of $\langle x^2 - y^2\rangle \subset \Bbbk[x,y]$. What degree must each monomial in $F_0 = R$ have for this resolution to be graded under the standard grading?

(e) The grading of F_0 in the previous problem is called a *shifted grading*, where we effectively "translate" the grading by some amount. Notationally, we write $R(a)$ to indicate that $\deg(x^b) + a$ is the original degree of $x^b \in R$. Determine the value of a so that

$$0 \longleftarrow \langle x^2 - y^2\rangle \longleftarrow R(a) \longleftarrow 0$$

is a (standard) graded free resolution (be careful!).

(f) Returning again to the free resolutions constructed before Problem (D1), identify the appropriate graded shift of each **summand** of each F_i, and verify that with those choices, the resolutions are graded. Note that for finitely graded resolutions, each shift value a will be a vector!

(g) We are now ready to examine minimality. A graded free resolution is *minimal* if every nonzero entry of each matrix is nonconstant. Determine which free resolutions constructed before Problem (D1) are minimal (the answer should not be surprising).

(h) Given a graded module M, the *graded Betti number* $\beta_{i,a}(M)$ equals the number of summands of $R(-a)$ appearing in F_i in a minimal graded free resolution for M. Find all nonzero Betti numbers of the modules I, J, R/I and R/J using the free resolutions from before Problem (D1).

It turns out that every graded module M has a **minimal graded** free resolution, and that any two minimal free resolutions for M are *isomorphic as graded free resolutions* (a notion we will not define here). This has numerous important consequences; for instance, the Betti elements of a numerical semigroup S with defining ideal I_S are precisely the values of a such that $\beta_{1,a}(R/I_S) > 0$. This can be seen in the last resolution before Problem (D1), where J is the defining ideal for $S = \langle 6, 9, 20 \rangle$, and $F_1 = R(-18) \oplus R(-60)$ contains some all-too-familiar values. More generally, the uniqueness of minimal free resolutions is what makes Betti elements (and more generally, graded Betti numbers) well-defined.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each of the following rings R and R-modules M, find a minimal graded free resolution of M, and use it to find the graded Betti numbers of M.

(a) $R = \Bbbk[x, y]$ and $M = R/\langle x^3, y^3, x^2y - xy^2\rangle$, under the standard grading.

(b) $R = \Bbbk[x, y, z]/\langle xy, xz, yz\rangle$ and $M = R/\langle x\rangle$, under the fine grading.

(H2) Suppose $I \subset R = \Bbbk[x, y]$ is a monomial ideal. Obtain a minimal free resolution for R/I, and characterize the (finely) graded Betti numbers in terms of the staircase diagram of I.

(H3) Suppose $I = \langle x_1, \ldots, x_k\rangle \subset R = \Bbbk[x_1, \ldots, x_k]$. Obtain a minimal free resolution for R/I, and find the (finely) graded Betti numbers.

(H4) Fix a field $\Bbbk$ and an exact sequence

$$0 \longleftarrow V_0 \longleftarrow V_1 \longleftarrow \cdots \longleftarrow V_\ell \longleftarrow 0$$

of finite dimensional vector spaces over $\Bbbk$. Prove that

$$\sum_{i=0}^{\ell} (-1)^i \dim(V_i) = 0.$$