Spring 2020, Math 621: Week 13 Problem Set
Due: Friday, May 8th, 2020
The Hilbert Syzygy Theorem

Warmup and Discussion problems. The problems below should be worked on in groups, but will not be submitted for credit. Only submit the homework problems at the end of this document. Try to read up through the start of the first discussion problem, and complete the warmup problems, prior to starting with your group. The content included covers enough material to complete the assigned problems, but if you are interested in further reading, I suggest Chapters 5 and 6 of *Using Algebraic Geometry* by David Cox, John Little, and Donal O'Shea.

This week, we will prove the second big theorem of Hilbert (the Syzygy Theorem), as well as the "Big" Hilbert's Theorem. For reference,

$$0 \longleftarrow I \xleftarrow{\begin{bmatrix} x^4y & x^2y^2 & xy^3 \end{bmatrix}} R^3 \xleftarrow{\begin{bmatrix} y & 0 \\ -x^2 & y \\ 0 & -x \end{bmatrix}} R^2 \longleftarrow 0.$$

is a (minimal) free resolution for $I = \langle x^4y, x^2y^2, xy^3\rangle \subset R = \Bbbk[x,y]$, borrowed from last week. Notice that $\ker\varphi_0 = \langle y\mathbf{e}_1 - x^2\mathbf{e}_2, y\mathbf{e}_2 - x\mathbf{e}_3\rangle$. It is relatively easy to show "$\supseteq$" holds, but the reverse containment need not be clear (and is even less clear in more complicated examples). Today, we will also see Schreyer's Theorem, which helps us ensure no generators are missing.

(D1) *The Hilbert Syzygy Theorem.* Let $R = \Bbbk[x_1,\ldots,x_k]$. The goal of this problem is to prove the following. Our proof will use Gröbner bases in an essential way.

Theorem (Hilbert Syzygy Theorem)**.** *Every finitely generated R-module M has a free resolution of length at most k, one that is graded if M is graded.*

(a) Given an R-module $M = \langle f_1,\ldots,f_r\rangle$, the *syzygy module* is a submodule of R^r given by

$$\mathrm{Syz}(f_1,\ldots,f_r) = \{(a_1,\ldots,a_r) \in R^r : a_1f_1 + \cdots + a_rf_r = 0\} \subset R^r.$$

Verify that $\mathrm{Syz}(f_1,\ldots,f_r) = \ker\varphi$, where $\varphi : R^r \to M$ is given by $\mathbf{e}_i \mapsto f_i$. Note that the syzygy module depends on the choice of **generating set** for M!

(b) Find $\mathrm{Syz}(x^4, x^2y^2, xy^3)$. Locate this module in the free resolution above Problem (D1).

(c) Fix a Gröbner basis $g_1,\ldots,g_r$ for M with respect to some term order $\prec$. Fixing i,j, Buchberger's criterion implies we can write

$$S(g_i,g_j) = a_1g_1 + \cdots + a_rg_r \qquad \text{for some} \qquad a_1,\ldots,a_r \in R^r.$$

Letting $L = \mathrm{lcm}(\mathrm{In}_\prec(g_i), \mathrm{In}_\prec(g_j))$, define

$$s_{ij} = \frac{L}{\mathrm{In}_\prec(g_i)}\mathbf{e}_i - \frac{L}{\mathrm{In}_\prec(g_j)}\mathbf{e}_j - a_1\mathbf{e}_1 - \cdots - a_r\mathbf{e}_r \in R^r.$$

Verify that $s_{ij} \in \mathrm{Syz}(g_1,\ldots,g_r)$.

(d) Let $J = \langle x^2 - y^2, xy^3 - x^2y, y^5 - x^3y\rangle$. Read Schreyer's Theorem, then use it to find a free resolution for J. Hint: the given generating set for J is a glex Gröbner basis.

Theorem (Schreyer)**.** *Given a Gröbner basis $g_1,\ldots,g_r$ for M under any term order $\prec$, the elements s_{ij} form a Gröbner basis for $\mathrm{Syz}(g_1,\ldots,g_r)$ under a new term order $\prec_G$ defined as follows: set $x^a\mathbf{e}_i \prec_G x^b\mathbf{e}_j$ whenever*

(i) $\mathrm{In}_\prec(x^ag_i) \prec \mathrm{In}_\prec(x^bg_j)$*; or*

(ii) $\mathrm{In}_\prec(x^ag_i) = \mathrm{In}_\prec(x^bg_j)$ *and* $i > j$.

In particular, $\mathrm{Syz}(g_1,\ldots,g_r) = \langle s_{ij} : 1 \le i,j \le r\rangle$.

(e) We will now prove the Hilbert Syzygy Theorem by arguing that the free resolution resulting from Schreyer's Theorem has length at most k.

(i) Argue that if $g_1, \ldots, g_r$ is a Gröbner basis for M and for some $m < k$, the variables $x_1, \ldots, x_m$ are absent from each $\mathrm{In}_\prec(g_i)$, then the variables $x_1, \ldots, x_{m+1}$ are absent from each $\mathrm{In}(s_{ij})$.

(ii) By induction and the previous part, decide what we can conclude for some $\ell < k$ about the initial terms in the resulting Gröbner basis $h_1, \ldots, h_r$ for $\ker \varphi_\ell$ in

$$0 \leftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_\ell} F_\ell.$$

Why can we not assume that $\ell = k - 1$?

(iii) Prove that $\mathrm{Syz}(h_1, \ldots, h_r) = 0$.

(iv) Conclude that the Hilbert syzygy theorem holds.

(f) Argue that if $M \subset R^n$ is a graded R-module, and $g_1, \ldots, g_r \in M$ form a homogeneous Gröbner basis for M, then each $s_{ij} \in \mathrm{Syz}$ is homogeneous as well. Conclude that Schreyer's Theorem yields a graded free resolution if M is graded.

Schreyer's Theorem yields, among other things, an algorithm to compute a free resolution of any finitely generated module. The resolution it produces need not be minimal (since Gröbner bases are often non-minimal generating sets), but a generalization of Schreyer's Theorem does yield (with a few extra steps) a Gröbner basis for $\mathrm{Syz}(f_1, \ldots, f_r)$ for **any** generators $f_1, \ldots, f_r$.

(W1) Find the Hilbert series of the ideal $I = \langle x^4y, x^2y^2, xy^3 \rangle$ (under the standard gradng) using its staircase diagram. Compare your numerator to the Betti numbers of I.

(D2) *Hilbert's Theorem.* We are finally ready to prove Hilbert's theorem, once and for all.

Theorem (Hilbert)**.** *Fix a finitely generated $\Bbbk$-algebra R, graded by $\mathbb{Z}^d_{\geq 0}$, with homogeneous generators $y_1, \ldots, y_k \in R$ of degrees $r_1, \ldots, r_k$, respectively, and fix a finitely generated graded R-module M. The Hilbert series of M has the form*

$$\mathrm{Hilb}(M; z) = \frac{h(z)}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}$$

for some polynomial $h(z)$ with integer coefficients.

(a) Argue that for some $\mathbb{Z}^d_{\geq 0}$-grading on $T = \Bbbk[x_1, \ldots, x_k]$ and some homogeneous ideal I, we have $R \cong T/I$ as graded rings. Note: you must specify I **and** the grading on T!

(b) Argue that M is a graded T-module.

(c) Argue that M has a graded free resolution of the form

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_k \longleftarrow 0$$

where each F_i is a free T-module.

(d) Use a homework problem from last week to argue that

$$\mathrm{Hilb}(M; z) = \sum_{i=0}^{k} (-1)^k \, \mathrm{Hilb}(F_i; z).$$

(e) Explain why

$$\mathrm{Hilb}(T; z) = \frac{1}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}.$$

Is it true that $\mathrm{Hilb}(R; z) = \mathrm{Hilb}(T; z)$?

(f) Conclude that Hilbert's theorem holds, and breathe a sign of relief knowing that your life is finally complete.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let $R = \Bbbk[x, y, z]$, $I = \langle x^3 - y^2, xy^6 - z^3 \rangle$, and $M = R/I$.

(a) Use Schreyer's Theorem (with the glex order on R) to compute a free resolution for M. Is the resolution you obtained minimal?

(b) Use the free resolution from part (a) to find the Hilbert series of M under the McNugget grading, defined by $\deg(x) = 6$, $\deg(y) = 9$, and $\deg(z) = 20$.

(H2) Once you have completed each of the following, it suffices to write "DONE" as your answer.

(a) Finish any remaining parts of the proof of the Hilbert Syzygy Theorem from Problem (D1) that your group did not complete in class.

(b) Finish any remaining parts of the proof of Hilbert's Theorem from Problem (D2) that your group did not complete in class.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If R is a Noetherian $\mathbb{Z}$-graded ring and M is a finitely generated graded R-module, then M has a free resolution with finite length.

(b) If $I \subset \Bbbk[x_1, \ldots, x_k]$ is a monomial ideal, then every syzygy module in a minimal free resolution for I is generated by monomials (i.e., elements of the form $x^a \mathbf{e}_i$).