Spring 2020, Math 621: Project Topics

The goal of each project is to learn about a topic not discussed in class. Throughout the semester, the following will be expected.

- Choose a topic. Please speak with me before making your decision, to ensure it is an appropriate level and so that we can narrow down a reasonable set of goals. You should choose a topic (and have it approved) no later than **Friday, March 27th**.
- Begin reading the agreed-upon background material. Plan to **meet at least twice** with me throughout the rest of the semester, to ensure that you are on track.
- Write (in LaTeX) a paper aimed at introducing your topic to fellow students, containing ample examples and explanations in addition to any theorems and proofs you give. Your writing should convey that you understand the intricacies of any proofs presented. Keep the following deadlines in mind as you proceed.
  - A rough draft of the paper will be due **Friday, May 1st** (one week before the last day of class). This will be peer reviewed by a fellow student in the following week.
  - The final paper will be due on **Tuesday, May 12th** (our "final exam" day).
- Optionally, give a 10-15 minute presentation introducing the main ideas of your topic. Presentations will take place (virtually) during the final exam slot at the semester's end. You should keep in mind your target audience and time constraints when deciding what and how to present. **No extra credit** will be offered for completing a presentation; your reward is the experience presenting mathematics.

  Note: although the presentation is optional, everyone is encouraged to present and should strongly consider doing so.
- Your final grade on the project will be determined by the content, quality, and completeness of your final writeup.

Given below are several project ideas. Many of the listed sources contain more material than is necessary for the project, so be sure to meet with me so we can set reasonable project goals. I am also open to projects not listed here, but you must run them by me before making a decision. Don't be afraid to ask questions at any point during the project!

Ideals

(1) *Edge ideals.* Edge ideals are a family of monomial ideals constructed from graphs. Many algebraic properties and constructions, such as free resolutions, can be obtained from combinatorial properties of their associated graph.

Source: *Edge ideals: algebraic and combinatorial properties* (S. Morey, R. Villarreal).

(2) *Binomial edge ideals.* Binomial edge ideals are a family of ideals constructed from graphs. Like their monomial counterparts, many algebraic properties can be obtained from combinatorial properties of their associated graph.

Source: *Binomial edge ideals: a survey* (S. Madani).

(3) *Hierarchical models and algebraic statistics.* Binomial ideals and semigroups arise in sampling problems from algebraic statistics. Each semigroup element corresopnds to a *fiber* of the model (essentially its set of factorizations), and binomial generators correspond to *Markov moves* within each fiber.

Source: *Algebraic algorithms for sampling from conditional distr.* (P. Diaconis, B. Sturmfels).

(4) *General binomial ideals.* In the paper that started it all, Eisenbud and Sturmfels introduce (general) binomial ideals, and use Gröbner bases to prove several foundational results.

Source: *Binomial ideals* (D. Eisenbud, B. Sturmfels).

(5) *Homology and Betti numbers.* Graded Betti numbers of monomial ideals can be computed using the homology of combinatorially constructed simplicial and polytopal complexes. This transforms the question of constructing a minimal free resolution from an algebraic one into a geometric/topological one.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 4.

Note: this project requires some familiarity with topology and/or homology.

(6) *Alexander duality of monomial ideals.* You may have noticed a visual phenomenon that can occur when staring at 3-variable staircase diagrams in which the staircase appears to turn "inside-out" with outward pointing corners suddently pointing inward and visa versa. This is encapsulated by the concept of *Alexander duality*, a key theorem from algebraic topology.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 5.

Note: this project requires some familiarity with topology and/or homology.

Semigroups and combinatorics

(7) *Hilbert functions and numerical semigroups.* A remarkable number of quantities from the realm of numerical semigroups turn out to exhibit eventually quasipolynomial behavior. Most proofs in the literature are "from first principles" and Hilbert's theorem is only recently starting to be utilized in this context.

Source: *On factorization invariants and Hilbert functions* (C. O'Neill).

(8) *Betti numbers and numerical semigroups.* The Betti numbers of a numerical semigroup algebra can be interpreted in terms of the factorizations of certain elements. Simplicial complexes naturally arise in this setting, as do Hilbert series.

Source: *Betti numbers for numerical semigroup rings* (D. Stamate).

(9) *Semigroups and matroids.* A recent paper considers a question involving both semigroups and matroids. Gröbner bases also make an appearance.

Source: *The monoid of monotone functions on a poset and arithmetic multiplicities for uniform matroids* (W. Bruns, P. García-Sánchez, L. Moci).

Note: this project requires some familiarity with matroids (e.g., enrollment in the other section of Math 621 this semester).

Polytopes

(10) *Matroid polytopes.* Matroids encapsulate the combinatorics of linear independence, and have a knack for arising in unexpected places. Given a matroid M, one can construct a polytope P with one vertex for each basis of M, and the combinatorial properties of P are closely connected to those of M (e.g., the edges of P correspond to basis exchanges, and the facet inequalities of P are obtained from the rank function of M).

Source: *Matroid polytopes and their volumes* (F. Ardila, C. Benedetti, J. Doker).

Note: this project requires some familiarity with matroids (e.g., enrollment in the other section of Math 621 this semester).

(11) *The Dehn-Sommerville relations.* The *face numbers* $f_0, f_1, \dots, f_d$ of a polytope P count the number of faces of P of dimension $0, 1, \dots, d$, respectively. If P is *simple* (meaning every vertex of P lies in exactly d edges, the smallest number geometrically possible), then the face numbers of P satisfy a collection of equations called the *Dehn-Sommerville relations.* These relations arise frequently in the study of Ehrhart functions of simple polytopes.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 5.