Spring 2021, Math 522: Problem Set 6
Due: Thursday, March 11th, 2021
Arithmetic Functions

(D1) *Formulas for $d(n)$ and $\sigma(n)$.* Let $d(n)$ denote the number of positive divisors of n, and let $\sigma(n)$ denote the sum of the positive divisors of n.

(a) Find $d(n)$, and $\sigma(n)$ for $n \leq 10$ and $n = 42$.

(b) Find a formula for $d(p)$ and for $\sigma(p)$ when p is prime.

(c) Find a formula for $d(p^r)$ and $\sigma(p^r)$ when p is prime and $r \geq 1$. Write your formula for $\sigma(p^r)$ as a fraction with denominator $p - 1$.

(d) We will prove in Problem (D2) that $d(ab) = d(a)d(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$ whenever $\gcd(a, b) = 1$. Use this and your above to derive formulas for $d(n)$ and $\sigma(n)$ in terms of the prime factorization $n = p_1^{r_1} \cdots p_k^{r_k}$.

(D2) *Multiplicative functions.* The goal for this problem is to prove that $d(n)$ and $\sigma(n)$ are multiplicative on relatively prime integers.

In what follows, let $D_n = \{d > 0 : d \mid n\}$ denote the set of positive divisors of n.

(a) Find D_4, D_{15}, and D_{60}.

(b) Given two subsets $A, B \subset \mathbb{Z}$, define $A \cdot B = \{ab : a \in A, b \in B\}$ as the set of products of an element of A by an element of B.

Find $D_4 \cdot D_{15}$. What do you notice?

(c) Suppose $\gcd(a, b) = 1$. Prove that if $d \mid ab$, then $d = a'b'$ for some $a' \mid a$ and $b' \mid b$.

(d) Argue that in the previous part, the integers a' and b' are **unique**.

(e) Are either of the previous 2 parts true if the hypothesis $\gcd(a, b) = 1$ is dropped?

(f) Conclude that if $\gcd(a, b) = 1$, then $D_a \cdot D_b = D_{ab}$ and $|D_{ab}| = |D_a||D_b|$.

(g) Conclude that if $\gcd(a, b) = 1$, then $d(ab) = d(a)d(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) Find $\phi(441)$ without using a calculator.
Hint: $441 = 3^2 7^2$.

(H2) Locate infinitely many integers n such that $10 \mid \phi(n)$.

(H3) Every $n \geq 1$ satisfies $d(n) < 2\sqrt{n}$.

(H4) (a) Prove that $n \mid (\phi(n)\sigma(n) + 1)$ if n is prime.
(b) Prove that divisibility fails to hold if $p^2 \mid n$ for some prime p.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) In this problem, we will prove that $\phi(n)$ is multiplicative on relatively prime integers, as was shown for both $d(n)$ and $\sigma(n)$ in discussion.

Let $\mathbb{Z}_n^* = \{[a]_n : \gcd(a, n) = 1\}$ denote the set of units in $\mathbb{Z}_n$, and consider the function $f : \mathbb{Z}_{nm}^* \to \mathbb{Z}_n^* \times \mathbb{Z}_m^*$ given by $f([a]_{nm}) = ([a]_n, [a]_m)$.

(a) Prove that f is well-defined, that is, if $[a]_{nm} = [b]_{nm}$, then $f([a]_{nm}) = f([b]_{nm})$.
(b) Prove that if $\gcd(n, m) = 1$, then f is one-to-one.
(c) Use the Chinese Remainder Theorem to prove that if $\gcd(n, m) = 1$, then f is onto.
(d) Use the previous parts to conclude that if $\gcd(n, m) = 1$, then $\phi(nm) = \phi(n)\phi(m)$.