

Spring 2021, Math 522: Problem Set 7
Due: Thursday, March 18th, 2021
The Möbius Function

(D1) *Möbius inversion formula.* We will be using, without proof for the moment, the following.

Theorem (Möbius inversion formula). *If $f, g : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfy*

$$f(n) = \sum_{d|n} g(d)$$

for every $n \in \mathbb{Z}$, then

$$g(n) = \sum_{d|n} \mu(d)f(n/d).$$

(a) For each $n = 1, \dots, 10$, find

$$\sum_{d|n} \mu(d)d\left(\frac{n}{d}\right).$$

Formulate a conjecture for general n . (No proof needed yet!)

(b) Use the Möbius inversion formula to prove your conjecture.

Hint: pick the functions f and g so that the conclusion of the Möbius inversion formula matches the equality you wish to prove.

(c) Use the Möbius inversion formula to find a formula (in terms of n) for

$$\sum_{d|n} \mu(d)\sigma\left(\frac{n}{d}\right).$$

Hint: begin by finding this sum for $n = 1, \dots, 10$.

(d) In class last week, we proved the first equality in

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

Use the Möbius inversion formula to prove the second equality.

(D2) *Proving the Möbius inversion formula.* The majority of the proof comes from carefully examining the following algebraic manipulation, which we will justify in several steps.

$$\begin{aligned} \sum_{d|n} \mu(d)f(n/d) &= \sum_{\substack{d, d' \leq n \\ d \cdot d' = n}} \mu(d)f(d') = \sum_{\substack{d, d' \leq n \\ d \cdot d' = n}} \mu(d) \sum_{e|d'} g(e) \\ &= \sum_{\substack{d, e, h \leq n \\ d \cdot e \cdot h = n}} \mu(d)g(e) = \sum_{\substack{e, h' \leq n \\ e \cdot h' = n}} g(e) \sum_{d|h'} \mu(d). \end{aligned}$$

- (a) For $n = 12$, write out each step **without** sigma-sums. Verify each equality in this case.
 (b) Give a thorough written justification of each step of the above algebra.
 (c) Explain why the final expression above equals $g(n)$, thereby completing the proof.
 (d) Using a similar argument, prove the converse of the Möbius inversion formula (that is, if the second equality holds for all n , then so does the first).

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) Draw the divisibility poset of $n = 441$. Try your best to obtain the most visually appealing drawing you can (this may take several attempts). Write the value $\mu(d)$ next to each element d .

Hint: $441 = 3^2 7^2$.

(H2) Prove that there are infinitely many integers n such that $\phi(n)$ is a perfect square.

(H3) (a) Prove that if $a \mid b$ then $\phi(a) \mid \phi(b)$.

(b) Given $n \geq 2$, conjecture a formula for

$$\sum_{d \mid n} (\mu(d))^2 \frac{\phi(n)}{\phi(d)}.$$

Demonstrate your conjecture holds for at least 5 values of n . How is part (a) relevant to this problem?

Note: you are **not** required to prove your conjecture.

(H4) Prove that for all $n \geq 2$,

$$\sum_{d \mid n} \mu(d) \phi(d) = \prod_{p \mid n} (2 - p).$$

Hint: induct on the number of distinct primes in the factorization of n .

(H5) Determine whether each of the following is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If $a \mid b$ and $\phi(a) = \phi(b)$, then $a = b$.

(b) For all $n \geq 2$, the quantity

$$\sum_{d \mid n} \frac{\phi(d) + 1}{d}$$

is an integer.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove your formula from Problem (H3)(b).

(C2) Given a finite poset (P, \preceq) , a function $f : P \rightarrow \mathbb{Z}$ is called *order preserving* if $p \preceq q$ implies $f(p) \leq f(q)$. Let $L_P(t)$ denote the number of order preserving functions $P \rightarrow \{1, \dots, t\}$.

Prove that for any poset (P, \preceq) , the function $L_P(t)$ is a polynomial in t of degree $|P|$.