Spring 2021, Math 522: Problem Set 9
Due: Thursday, April 8th, 2021
Locating Large Primes

(D1) *Treading water in hard-problem infested seas.* The following are claims someone might walk up to you on the street and make. Use intuition about prime numbers, as discussed in class on Tuesday, to determine whether the statement is likely true or likely false. Give a detailed justification of your position.

After completing the first 3 parts, and after completing all parts, verify with Brittney or Chris that you settled on the correct truth value for each.

(a) For every $n \geq 2$, each equivalence class modulo n contains infinitely many primes.

(b) For every $n \geq 2$, each equivalence class modulo n that contains at least one prime contains infinitely many primes.

(c) For every prime $p > 1000$, the number $p^p + 2$ is also prime.

(d) For every $k \geq 1$, the interval $[100k, 100k + 99]$ (i.e., each range 100-199, 200-299, etc.) contains at least one prime.

(e) There exist arbitrarily large gaps between sequential primes.

(f) There exist infinitely many integers n such that n, $n + 2$, and $n + 4$ are all prime (we might want to call this the "triplet prime conjecture").

(D2) *And now, some proofs.* With the exception of part (c), every statement in Problem (D1) **can** be proven (or disproven) with the tools we obtained on Tuesday. Do so now.

Be sure you have verified with Brittney or Chris that you settled on the correct truth value for each before attempting a proof!

(D3) *A weaker version of the Prime Number Theorem.* The Prime Number Theorem states

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = 1.$$

In this problem, we will prove a weaker result, namely that

$$\pi(n) \geq \log_2(\log_2(n))$$

for every $n \geq 2$.

(a) For each $n \geq 1$, let $a_n = 2^{2^n} + 1$. Prove that if $n < m$, then $a_n \mid (a_m - 2)$

(b) Use part (a) to prove that if $n \neq m$, then $\gcd(a_n, a_m) = 1$.

(c) Conclude that there are at least n primes p such that $p \leq a_n$.

(d) Conclude that $\pi(n) \geq \log_2(\log_2(n))$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) Define the sequence $a_1, a_2, \ldots$ in the following way: let $a_1 = 2$, and for each $n \geq 2$, let a_n be the smallest integer such that $\gcd(a_i, a_n) = 1$ for all $i < n$. Use induction on n to prove that a_n coincides with the n-th prime.

(H2) ****** EXCISED ******

(H3) For this problem, you may **not** use Dirichlet's theorem.

- (a) Prove that infinitely many primes p satisfy $p \equiv 3 \bmod 4$.
  Hint: assume $p_1, \ldots, p_k$ are all such primes, and consider $n = 4p_1 \cdots p_k - 1$.
- (b) Use Problem (H2) to prove that infinitely many primes p satisfy $p \equiv 1 \bmod 4$.
  Hint: assume $p_1, \ldots, p_k$ are all such primes, and consider $n = 4p_1^2 \cdots p_k^2 + 1$.
  You may find the following fact helpful (from Problem (H5)(b) on Homework Set 8): given $n \geq 1$, any odd prime p dividing $n^2 + 1$ must satisfy $p \equiv 1 \bmod 4$.

(H4) Determine whether each of the following is true or false. Prove each true statement, and give a counterexample for each false statement.

- (a) For each $k \geq 1$, there exists an n such that $n, n+1, \ldots, n+k$ are all composite.
- (b) For every $n \geq 1$, the integer $n^2 - n + 41$ is prime.
- (c) If n is composite, then $2^n - 1$ is also composite.
- (d) If n is prime, then $2^n - 1$ is also prime.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Read the proof of Tchebychev's Theorem in Andrews 8.2, and write a "roadmap" for the proof (i.e., a summary of the main steps, without including any algebraic details).