Spring 2021, Math 522: Problem Set 11
Due: Thursday, April 22nd, 2021
Cyclotomic Polynomials

(D1) *Finding cyclotomic polynomials.* Factor $x^n - 1$ as a product of cyclotomic polynomials for each of the following values of n. Identify each factor as $\Phi_d(x)$ for some $d \mid n$.

Hint: you may find the following formulas useful.

$$a^2 - 1 = (a+1)(a-1), \qquad a^3 - 1 = (a-1)(a^2+a+1), \qquad a^3 + 1 = (a+1)(a^2 - a + 1)$$

(a) $n = 3$
(b) $n = 9$
(c) $n = 8$
(d) $n = 18$ (hint: $\Phi_{18}(x)$ has 3 nonzero terms)
(e) $n = 24$ (hint: $\Phi_{24}(x)$ has 3 nonzero terms)

(D2) *Some general formulas.*

(a) Find $\Phi_p(x)$ for p prime.
(b) Find $\Phi_n(x)$ when $n = 2^k$ for some $k \geq 1$. Prove your formula holds.
Hint: use induction on k.
(c) Find a formula for $\Phi_n(0)$ that holds for every $n \geq 2$. Prove that your formula holds.
Hint: consider how $x^n - 1$ factors.
(d) Compute $\Phi_n(-1)$ for each odd $n \leq 10$.
(e) Conjecture and prove a general formula for $\Phi_n(-1)$ when $n > 1$ is odd.
(f) Find $\Phi_n(x)$ when $n = 3^k$ for some $k \geq 1$. Prove your formula holds.

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) Factor $x^{20} - 1$ as a product of cyclotomic polynomials. Identify each factor as $\Phi_d(x)$ for some $d \mid 20$.

(H2) Show that if $n \geq 3$ is odd, then $\Phi_{2n}(x) = \Phi_n(-x)$.

(H3) Let $N = \Phi(n)$. Prove that the coefficients of $\Phi_n(x)$ are symmetric (that is, if we write

$$\Phi_n(x) = a_N x^N + a_{N-1} x^{N-1} + \cdots + a_1 x + a_0,$$

then $a_i = a_{N-i}$ for each i).

Hint: start by showing that $\Phi_n(x)$ and $x^N \Phi_n(1/x)$ have the same complex roots.

(H4) Find a formula for $\Phi_n(1)$ in terms of n. Prove your formula holds.

Hint: your formula will likely depend on how many distinct prime factors n has.

(H5) Determine whether each of the following is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) For every $n \geq 3$, we have $\Phi_{2n}(x) = \Phi_n(-x)$.

(b) The roots of $x^n - 1$ form the vertices of a regular n-gon.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Find a formula for $\Phi_n(-1)$ in terms of n.