Spring 2021, Math 522: Problem Set 14
Due: Thursday, May 6th, 2021
Additive Number Theory

(D1) *Minimal zero-sum sequences.*

(a) Find all 8 minimal zero-sum sequences over $\mathbb{Z}_7$ of length 3.

(b) Find all 15 minimal zero-sum sequences over $\mathbb{Z}_5$.

(c) Determine the length of the longest minimal zero-sum sequence over $\mathbb{Z}$ that only uses the letters $[-2], [-1], [0], [1], [2]$.

(d) Determine the length of the longest minimal zero-sum sequence over $\mathbb{Z}$ that only uses the letters $[-3], [-2], [-1], [0], [1], [2], [3]$.

(D2) *Arithmetic congruence monoids.* If $a, b \geq 1$ satisfy $a^2 \equiv a \bmod b$, then the set

$$M_{a,b} = \{1\} \cup \{n \geq 1 : n \equiv a \bmod b\}$$

is known as an *arithmetical congruence monoid.*

(a) Verify that $M_{4,6}$ is an arithmetic congruence monoid, as is $M_{1,b}$ for any $b \geq 1$.

(b) Write down the first 10 elements of $M_{1,4}$. Have each member of your group pick 2 elements from this list to mutiply together, and verify the product also lies in $M_{1,4}$. Do the same for $M_{4,6}$.

(c) Prove that for any $n, m \in M_{a,b}$, we have $nm \in M_{a,b}$.

(d) We say $n \in M_{a,b}$ is *irreducible* if $n > 1$ and we cannot write $n = m_1 m_2$ for some $m_1, m_2 \in M_{a,b}$ with $m_1, m_2 > 1$.

Argue that 5, 9, 13, 21, and 49 are irreducible in $M_{1,4}$.

(e) Argue that $4 \cdot 7^r$ is irreducible in $M_{4,6}$ for any $r \geq 1$.

(f) A *factorization* of an element $n \in M_{a,b}$ is an expression of the form

$$n = m_1 \cdots m_k$$

such that each $m_i \in M_{a,b}$ is irreducible.

Find 2 factorizations of $441 \in M_{1,4}$ that are distinct (up to reordering).

Hint: $441 = 3^2 7^2$.

(g) Find all 3 factorizations of 34000 in $M_{4,6}$.

Hint: $34000 = 2^4 \cdot 5^3 \cdot 17$.

(h) Find all 8 factorizations of 442000 in $M_{4,6}$.

Hint: $442000 = 2^4 \cdot 5^3 \cdot 13 \cdot 17$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) (a) Find all 20 minimal zero-sum sequences over $\mathbb{Z}_6$.

(b) Find a formula, in terms of n, for the number of minimal zero-sum sequences over $\mathbb{Z}_n$ of length 2, and prove your formula holds.

(H2) (a) Prove that if $m \in M_{1,4}$ is irreducible, then either (i) m is prime in $\mathbb{Z}$, or (ii) $m = p_1 p_2$ for primes p_1, p_2 satisfying $p_1, p_2 \equiv 3 \bmod 4$.

(b) Locate an irreducible element of $M_{1,7}$ that is a product of exactly 6 primes in $\mathbb{Z}$.

Note: it turns out this is the "longest" an irreducible in $M_{1,7}$ can be; that is, any irreducible in $M_{1,7}$ is a product of at most 5 primes in $\mathbb{Z}$.

(c) Locate an irreducible element of $M_{1,101}$ that is a product of exactly 100 primes in $\mathbb{Z}$.

Note: it turns out this is, again, the "longest" an irreducible in $M_{1,101}$ can be.

(H3) Determine whether each of the following is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) There is an irreducible element of $M_{6,10}$ that is a product of exactly 19 primes in $\mathbb{Z}$.

(b) No irreducible element of $M_{1,6}$ is a product of exactly 4 primes in $\mathbb{Z}$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: for each $b \geq 1$, there exists $\ell \geq 1$ such that for any irreducible $m \in M_{1,b}$, the prime factorization of m in $\mathbb{Z}$ has at most ℓ primes.