Spring 2021, Math 621: Problem Set 1
Due: Thursday, February 4th, 2021
Point Set Topology

(D1) *(Topological) Manifolds.* Recall that a d-dimensional topological manifold (a d-manifold) is a space that is locally homeomorphic to $\mathbb{R}^d$.

(a) Recall that $S^d = \{p \in \mathbb{R}^{d+1} : \|p\| = 1\}$ is the (hollow) unit sphere in $\mathbb{R}^{d+1}$.

(i) Let $e_{d+1} \in S^d$ be the "north pole". Locate a homeomorphism $\mathbb{R}^d \to S^d \setminus \{e_{d+1}\}$. Hint: start by drawing a picture for $d = 1$, with $\mathbb{R}^1$ depicted as the x-axis in $\mathbb{R}^2$.

(ii) Locate a second "patch" in S^d to conclude that S^d is a topological manifold.

(iii) Locate a covering of S^d by exactly $2(d+1)$ patches, each homeomorphic to $\mathbb{R}^d$, so that no patch can be omitted while still covering S^d.
Hint: start with $d = 1$, and remember any open interval is homeomorphic to $\mathbb{R}$.

(b) Fix a d_1-manifold T_1 and a d_2-manifold T_2. Prove that $T_1 \times T_2$ is a manifold, and determine its dimension.

(D2) *Projective space.* In this problem, we will construct *real projective space* $\mathbb{RP}^d$.

(a) Fix an equivalence relation $\sim$ on S^d. Is $S^d/\!\sim$ necessarily a manifold?

(b) Let $\mathbb{RP}^d = S^d/\!\sim$, where $a, b \in S^d$ satisfy $a \sim b$ whenever $a = b$ or $a = -b$. Prove $\mathbb{RP}^d$ is a manifold. Hint: use Problem (D1)(a).

(c) Prove that $\mathbb{RP}^1$ is homeomorphic to S^1. Is the same true for $\mathbb{RP}^2$ and S^2?

(d) Let $T = (\mathbb{R}^{d+1} \setminus \{0\})/\!\sim$, where $a, b \in \mathbb{R}^{d+1}$ with $a, b \neq 0$ satisfy $a \sim b$ whenever $a = \lambda b$ for some $\lambda \in \mathbb{R}$. In what way can we think of T as the set of lines in $\mathbb{R}^{d+1}$?

(e) Prove that T is homeomorphic to $\mathbb{RP}^d$.

(f) The previous part yields a continuous surjection $\mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{RP}^d$. Can this map be extended to a continuous map $\mathbb{R}^{d+1} \to \mathbb{RP}^d$?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Determine whether each of the following is a topological manifold. Prove your claims.

(a) A (hollow) unit square in $\mathbb{R}^2$.

(b) A d-manifold T for $d \geq 1$ with a single point $p \in T$ removed.

(c) The set $T_1 \cap T_2 \subset \mathbb{R}^3$ under the subspace topology, where $T_1, T_2 \subset \mathbb{R}^3$ are 2-manifolds with nonempty intersection.

(H2) Fix a topological space T. A subset $C \subseteq T$ is *compact* if every open cover of C has a finite subcover; that is, whenever $C \subseteq \bigcup_{i \in \mathcal{I}} U_i$ for some collection of open sets $U_i \subseteq T$, there must exist $V_1, \ldots, V_k \in \{U_i : i \in \mathcal{I}\}$ such that $C \subseteq \bigcup_{i=1}^{k} V_i$. Note that the index set $\mathcal{I}$ could have any cardinality; it could be infinite, even uncountable, or it could be finite.

(a) Using only the definition, prove that the set $C = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subset \mathbb{R}$ is compact.

(b) Prove that if $C \subseteq \mathbb{R}^d$ is compact, then C is closed and bounded.
Note: this is actually an "if and only if" but the converse is substantially harder!

(H3) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) Under a continuous map, the image of any compact set is compact.

(b) Under a continuous map, the preimage of any compact set is compact.

(c) If $f(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d]$ is a polynomial with integer coefficients, then the set

$$T = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : f(a_1, \ldots, a_d) = 0\}$$

(called the *graph* of f) is a topological $(d-1)$-manifold.