Spring 2021, Math 621: Problem Set 2
Due: Thursday, February 11th, 2021
Simplicial Complexes and Cell Complexes

(D1) *Simplicial Complexes.*

(a) Draw each of the following simplicial complexes, and determine which familiar space each is homeomorphic to.

(i) $\{123, 124, 134, 234\}$

(ii) $\{123, 134, 145, 125\}$

(b) Find a triangulation of the Möbius band (i.e. a 2-dimensional strip of ribbon with a single twist). Do this in two different ways (e.g., with a different number of triangles).

(c) The *Euler characteristic* of a simplicial complex Δ is the alternating sum

$$\chi(\Delta) = f_0 - f_1 + f_2 - f_3 + \cdots$$

where f_i equals the number of i-dimensional faces of Δ. Compute the Euler characteristic of both triangulations of the Möbius band in the previous part.

(D2) *Topology of group actions.*

(a) Suppose a group G also happens to be a topological space. Given a normal subgroup H, determine a natural topological structure for H, and for the quotient group G/H.

(b) A group G that is also a topological space is called a *topological group* if the group operation, viewed as a map $G \times G \to G$, is continuous.

Verify that $\mathbb{R}^d$ is a topological group.

(c) What familiar topological space is the quotient group $\mathbb{R}^2/\mathbb{Z}^2$ (under the topology induced by $\mathbb{R}^2$)?

(d) A *group action* is a (not necesssarily topological) group G and a set T together with an operation $G \times T \to T$ such that $g_1(g_2t) = (g_1g_2)t$ for any $g_1, g_2 \in G$ and $t \in T$. The *orbit* of $t \in T$ is the set

$$Gt = \{gt : g \in G\} \subset T.$$

It is known that the set of orbits, denoted T/G, is a partition of T.

Let $T = \mathbb{R}^d \setminus \{\vec{0}\}$, $G = (\mathbb{R}_{>0}, \cdot)$, and $G' = (\mathbb{R} \setminus \{0\}, \cdot)$. Define a natural action of G on T, and determine which familiar space is T/G. Do the same with G'.

(e) Consider S^1 as the unit circle in $\mathbb{R}^2$, and define an action of the symmetric group S_2 on S^1 by coordinate permutation. Find a cell decomposition of S^1/S_2 by first finding a cell decomposition of S^1 such that the image of any cell under the action of S_2 is again a cell (i.e., the action of S_2 on S^1 induces a permutation on cells).

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Draw each of the following simplicial complexes, and determine which familiar space each is homeomorphic to.

(a) $\Delta = \{123, 134, 145, 125, 236, 346, 456, 256\}$

(b) $\Delta = \{12, 13, 14, 15, 16, 17, 23, 45, 67\}$

(H2) Recall $S^1 \times S^1$ is a torus and $S^1 \times D^2$ is a solid torus. Consider S^3 as

$$S^3 = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\}.$$

that is, the unit sphere in $\mathbb{C}^2$. Fix a real number a satisfying $0 < a < 1$, and let

$$\begin{aligned} T(a) &= \{(x, y) \in S^3 : |x|^2 = a\}, \\ U_1(a) &= \{(x, y) \in S^3 : |x|^2 \leq a\}, \text{ and} \\ U_2(a) &= \{(x, y) \in S^3 : |x|^2 \geq a\}. \end{aligned}$$

Prove that $T(a)$ is a torus and that $U_1(a)$ and $U_2(a)$ are solid tori with intersection $T(a)$. Deduce the homeomorphism

$$S^3 \cong (U_1(a) \amalg U_2(a))/\sim$$

where the equivalence relation $\sim$ identifies $x \in U_1(a)$ with $y \in U_2(a)$ iff $x = y \in T(a)$.

(H3) A *group action* is a (not necesssarily topological) group G and a set T together with an operation $G \times T \to T$ such that $g_1(g_2t) = (g_1g_2)t$ for any $g_1, g_2 \in G$ and $t \in T$. The *orbit* of $t \in T$ is the set

$$Gt = \{gt : g \in G\} \subset T.$$

It is known that the set of orbits, denoted T/G, is a partition of T.

(a) Let $T = \mathbb{R}^d \setminus \{\vec{0}\}$, $G = (\mathbb{R}_{>0}, \cdot)$, and $G' = (\mathbb{R} \setminus \{0\}, \cdot)$. Define a natural action of G on T, and determine which familiar space is T/G. Do the same with G'.

(b) Consider S^2 as the unit sphere in $\mathbb{R}^3$, on which the symmetric group S_3 acts by coordinate permutation. Find a cell decomposition of S^2 that induces a cell decomposition on S^2/S_3. Try to use as few cells as possible.

(H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) Given triangulations Δ_1 and Δ_2 of topological spaces T_1 and T_2, respectively, there exists a triangulation of $T_1 \times T_2$ whose vertices are ordered pairs (x, y) where x and y are vertices of Δ_1 and Δ_2, respectively.

(b) If G is a group acting on a (topological) d-manifold X, then X/G is also a manifold.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Find a triangulation of the torus with the smallest number of triangles.