Spring 2021, Math 621: Problem Set 3
Due: Thursday, February 18th, 2021
Homotopy Equivalence (Week 1)

(D1) *Homotopy equivalence of spaces.*

(a) Partition the 26 capital letters of the English alphabet based on homotopy equivalence. Within each equivalence class, further partition based on which are homeomorphic to one another. Lastly, decide which are manifolds.

(b) Each of the following space is homotopy equivalent to a wedge sum of spheres of various dimensions. For each, find this wedge sum. You are **not** required to prove your claims.

(i) The space T obtained by identifying the north and south poles of S^d.

(ii) The space T given by the union of a torus with n meridian disks "inside" the torus (each disk is where/how you might cut a donut to split it amongst n friends).

(c) Fix a space X. Obtain an explicit deformation retraction of the *cone*

$$CX = (X \times I)/(X \times \{0\})$$

onto a point.

(D2) *New spaces from old.*

(a) Recall that the join $X * Y$ of two topological spaces X and Y is the space $X \times Y \times I$ with each subset of the form $X \times \{y\} \times \{0\}$ or $\{x\} \times Y \times \{1\}$ identified to a point. It turns out that if X and Y have triangulations Δ_1 and Δ_2, then $X * Y$ has a triangulation

$$\Delta = \{F \cup F' : F \in \Delta_1, F' \in \Delta_2\}$$

whose vertex set is the (disjoint) union of the vertices of Δ_1 and Δ_2.
Obtain a triantulation of $S^1 * I$, and identify which familiar space it is.

(b) Define the *smash product* of pointed spaces (T_1, x_1) and (T_2, x_2) as the quotient

$$T_1 \wedge T_2 = (T_1 \times T_2)/(T_1 \vee T_2)$$

wherein $T_1 \vee T_2$ is identified with the subspace $(T_1 \times \{x_2\}) \cup (\{x_1\} \times T_2)$ of $T_1 \times T_2$.
Determine which familiar space is homeomorphic to $S^n \wedge S^m$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Define the *smash product* of pointed spaces (T_1, x_1) and (T_2, x_2) as the quotient

$$T_1 \wedge T_2 = (T_1 \times T_2)/(T_1 \vee T_2)$$

wherein $T_1 \vee T_2$ is identified with the subspace $(T_1 \times \{x_2\}) \cup (\{x_1\} \times T_2)$ of $T_1 \times T_2$. Determine which familiar space is homeomorphic to $S^n \wedge S^m$.

(H2) Construct a 2-dimensional simplicial complex (i.e., the largest faces are triangles) that deformation retracts onto both an annulus and a Möbius band.

(H3) Choose a point $p \in S^1 \times S^1$. Find an explicit deformation retraction of $(S^1 \times S^1) \setminus \{p\}$ (i.e., a punctured torus) onto $S^1 \vee S^1$.

(H4) Each of the following spaces is homotopy equivalent to a wedge sum of one or more familiar spaces. Find such an expression for each. You are **not** required to prove your claims, but you **are** required to provide clear descriptions of the deformation processes.

(a) $\mathbb{R}^2$ with the y-axis and one additional point removed

(b) S^d with a single point removed

(c) S^3 with its intersection with the xy-plane removed

(H5) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If $X \subset Y \subset Z$ are topological spaces such that Y deformation retracts onto X and Z deformation retracts onto Y, then Z deformation retracts onto X.

(b) If (X, x_0) and (Y, y_0) are pointed cell complexes and $(x_1, y_1) \in X \times Y$ with $x_0 \neq x_1$ and $y_0 \neq y_1$, then $(X \times Y) \setminus \{(x_1, y_1)\}$ deformation retracts onto $X \vee Y$, where

$$X \vee Y = \{(x, y) \in X \times Y : x = x_0 \text{ or } y = y_0\}$$

is viewed as a subspace of $X \times Y$.