Spring 2021, Math 621: Problem Set 4
Due: Thursday, February 25th, 2021
Homotopy Equivalence (Week 2)

(D1) *Contractible subspaces.* In what follows, suppose (X, A) is a CW-pair. In this problem, we will prove the 3-part theorem from the end of Tuesday's class.

(a) Prove the subspace $(X \times \{0\}) \cup (A \times I)$ is a deformation-retraction of $X \times I$.

(i) Describe a deformation retraction of $D^n \times I$ onto $(D^n \times \{0\}) \cup (\partial D^n \times I)$.

(ii) Describe a deformation retraction from $X^n \times I$ onto $(X^n \times \{0\}) \cup (A^n \times I)$.

(iii) Suppose $X = X^d$ for some $d \geq 0$ (that is, X is a finite dimensional cell complex). Describe a deformation retraction of $X \times I$ onto $(X \times \{0\}) \cup (A \times I)$.
Note: with a sufficiently careful argument, this can also be proven in the case where X is not finite dimensional.

(b) Prove that given $f_0 : X \to Y$, any homotopy $f_t : A \to Y$ that agrees with f_0 on A can be extended to $f_t : X \to Y$ (that is, (X, A) has the homotopy extension property).

(i) Construct a map $f : (X \times \{0\}) \cup (A \times I) \to Y$.

(ii) Let $h_t : X \times I \to X \times I$ denote the homotopy from part (a) with $h_0 = \mathrm{Id}_{X \times I}$ and $\mathrm{Im}(h_1) = (X \times \{0\}) \cup (A \times I)$. Use this to construct the extension $f_t : X \to Y$.

(c) Prove if A is contractible, the quotient map $q : X \to X/A$ is a homotopy equivalence.

(i) Fix a contraction $f_t : A \to A$. Use part (b) to obtain $f_t : X \to X$ with $f_0 = \mathrm{Id}_X$.

(ii) Define $\overline{f}_t : X/A \to X/A$ so the left diagram below commutes (i.e., $qf_t = \overline{f}_t q$).

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_t"] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_1"] \arrow[ur, "g"] & X/A \end{tikzcd}$$

(iii) Define a map $g : X/A \to X$ so that the right diagram above commutes.

(iv) Verify qg and gq are each *nullhomotopic* (i.e., homotopic to the identity map).

(D2) *Mapping cylinders.* Given a continuous map $f : X \to Y$, the *mapping cylinder* is the space

$$M_f = ((X \times I) \amalg Y)/{\sim}$$

where $(x, 1) \sim f(x)$ for each $x \in X$.

(a) Describe the mapping cylinder of the map $f : S^1 \to S^1 \times I$ taking S^1 onto the inner boundary of the annulus.

(b) Argue that M_f deformation retracts onto Y.

(c) Suppose f is a homotopy equivalence, with $g : Y \to X$ and homotopies $h_t : X \to X$ and $\ell_t : Y \to Y$ demonstrating $fg \simeq \mathrm{Id}_X$ and $gf \simeq \mathrm{Id}_Y$, respectively. We wish to prove the natural map $f' : X \hookrightarrow M_f$ whose image is $X \times \{0\}$ is also a homotopy equivalence.

(i) Construct the map $g' : M_f \to X$. Start by determining $g'(y)$ for $y \in Y$.

(ii) Locate a homotopy demonstrating $g'f' \simeq \mathrm{Id}_X$.

(iii) Find the homotopy $\ell'_t : M_f \to M_f$ demonstrating $f'g' \simeq \mathrm{Id}_{M_f}$.
Hint: intuitively, since the image of $f'g'$ is $X \times \{0\}$, we need ℓ'_t to expand along the interval I to Y, then out to cover Y. As such, ℓ'_t should do something different for $t \leq \frac{1}{2}$ than for $t \geq \frac{1}{2}$. Start by defining $\ell'_t(y)$ for $y \in Y$ (in both ranges of t), then define $\ell'_t(x, t')$.

(d) Why is h''_t not a deformation retraction onto $X \times \{0\}$?
Note: it turns out M_f does deformation retract onto $X \times \{0\}$; see Corollary 0.21.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that homotopy equivalence of topological spaces is an equivalence relation.

(H2) Prove that if $X \simeq X'$ and $Y \simeq Y'$, then $X \times Y \simeq X' \times Y'$.

(H3) Prove if A is contractible, the quotient map $q : X \to X/A$ is a homotopy equivalence. Use the following steps as a guide.

(i) Fix a contraction $f_t : A \to A$. Use the fact that (X, A) has the homotopy extension property to obtain $f_t : X \to X$ with $f_0 = \mathrm{Id}_X$.

(ii) Define $\overline{f}_t : X/A \to X/A$ so the left diagram below commutes (i.e., $qf_t = \overline{f}_t q$).

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_t"] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f}_1"] \arrow[ur, "g"] & X/A \end{tikzcd}$$

(iii) Define a map $g : X/A \to X$ so that the right diagram above commutes.

(iv) Verify qg and gq are each *nullhomotopic* (i.e., homotopic to the identity map).

(H4) Given a continuous map $f : X \to Y$, the *mapping cylinder* is the space

$$M_f = ((X \times I) \amalg Y)/\sim$$

where $(x, 1) \sim f(x)$ for each $x \in X$. It turns out that if f is a homotopy equivalence, then M_f deformation retracts onto both Y and $X \times \{0\} \cong X$.

Prove that M_f deformation retracts onto Y.

(H5) If X and Y are cell complexes, under what conditions (if any) on $f : X \to Y$ does the mapping cylinder M_f inherit a "natural" a cell structure from X and Y?

(H6) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) Any 1-dimensional cell complex with finitely many cells is homotopy equivalent to a wedge sum of circles.

(b) If X is a cell complex equal to the union of two contractible subcomplexes whose intersection is also contractible, then X is contractible.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose (X, A) is a CW-pair and that A is contractible in X (that is, the inclusion map $A \hookrightarrow X$ is homotopic to a constant map). Determine the homotopy type of X/A.

Hint: first find an example where A is not a contractible space but **is** constractible in X.