Spring 2021, Math 621: Problem Set 5
Due: Thursday, March 4th, 2021
The Fundamental Group

(D1) *Wild conjecture time.* For each of the following spaces T, conjecture which familiar group is the fundamental group.

(a) $T = \mathbb{RP}^2$

(b) $T = S^1 \times S^1$

(c) $T = S^1 \vee S^1$

(D2) *Product spaces.* The goal of this problem is to prove

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

for pointed spaces (X, x_0) and (Y, y_0).

(a) Using the natural projection maps $p : X \times Y \to X$ and $q : X \times Y \to Y$, construct a homomorphism $\pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(b) Argue this map is an isomorphism.

(D3) *Functoriality of the fundamental group.* Given pointed spaces (X, x_0) and (Y, y_0) and a continous map $f : X \to Y$ with $f(x_0) = y_0$, the map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, y_0) \\ [\rho] &\longmapsto [f \circ \rho] \end{aligned}$$

is a group homomorphism. Moreover, it turns out if f is a homotopy equivalence, then f_* is an isomorphism. The goal of this problem is to prove these facts.

(a) Let $S^1 = [0, 1]/(0 \sim 1)$. For $f : S^1 \to S^1 \times S^1$ given by $t \mapsto (t, t)$, find f_*.

(b) Verify f_* is well-defined (that is, if $[\rho] = [\rho']$, then $[f \circ \rho] = [f \circ \rho']$).

(c) Prove that f_* is a group homomorphism.

(d) (i) Find $\pi_1(\mathbb{R}^n \setminus \{\vec{0}\})$ for each $n \geq 1$.

(ii) Use the previous part to argue that $\mathbb{R}^n \not\cong \mathbb{R}^2$ whenever $n > 2$.

(e) Prove that if f is a homotopy equivalence, then f_* is an isomorphism.

(f) Prove or disprove: if f_* is an isomorphism, then f is a homotopy equivalence.

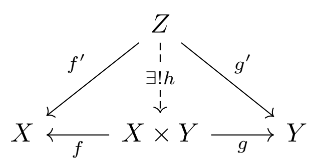
(g) Suppose $f : D^2 \to D^2$ has no fixed points. Define a map $g : D^2 \to S^1$ that sends $x \in D^2$ to the boundary point along the vector that points from $f(x)$ to x. What can be said about the image of g_*? What does this tell you about f?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$, locate a map $f : S^1 \to S^1$ such that $f_* = \varphi$.

(H2) Prove that if G is a topological group (that is, G is both a topological space and a group whose operation $G \times G \to G$ is continuous), then $\pi_1(G, e)$ is Abelian.

(H3) Recall that the *direct product* of two objects X, Y in a category $\mathcal{C}$ is the object $X \times Y$ together with morphisms $f : X \times Y \to X$ and $g : X \times Y \to Y$ satisfying the following universal property: given any object Z with maps $f' : Z \to X$ and $g' : Z \to Y$, there exist a unique map $h : Z \to X \times Y$ such that the diagram below commutes.



Prove that the category PTop of **pointed** topological spaces has products.

(H4) Complete Problem (D3) parts (b), (c), and (e). It suffices to write "DONE" as your answer.

(H5) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) Every homomorphism $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ has the form $\varphi = f_*$ for some $f : S^1 \times S^1 \to S^1$.

(b) If a continuous map $f : (X, x_0) \to (Y, y_0)$ is injective, then f_* is injective.

(c) If a continuous map $f : (X, x_0) \to (Y, y_0)$ is surjective, then f_* is surjective.

(d) If $f : X \to Y$ and $g : X \to Y$ satisfy $f_* = g_*$, then $f \simeq g$.