Spring 2021, Math 621: Problem Set 6
Due: Thursday, March 11th, 2021
The van Kampen Theorem

(D1) *Using The van Kampen Theorem.* Find the fundamental group of each of the following spaces. Decide if any two are isomorphic.

(a) The Klein bottle, presented as a square with opposite sides identified in the usual way.

(b) The Möbius strip with a 2-disk glued along their boundary edges.

(c) Two Möbius strips glues along their boundary edges.

(D2) *Fundamental groups of cell complexes.* Suppose X is a path-connected cell complex with finitely many cells (all claims in this problem also hold if X has countably many cells).

(a) Prove that the inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.

(b) Find $\pi_1(X, x_0)$ if $\dim X = 1$.

(c) Suppose X has a single 2-cell with gluing map $\phi : S^1 \to X^1$. Using the van Kampen Theorem, locate a generating set for the kernel of the map $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$. Hint: pick a path γ from x_0 to the image of the basepoint of S^1.

(d) Verify that choosing a different path δ in the previous part yields an identical kernel.

(e) Use the van Kampen theorem to give a method for obtaining $\pi_1(X, x_0)$.

(f) Compute $\pi_1(\mathbb{RP}^d, x_0)$ for each $n \geq 2$.

(g) For each of the following groups G, construct a cell complex X with $\pi_1(X) = G$.

(i) $G = \mathbb{Z}_n$.

(ii) $G = D_4$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Let $X \subset \mathbb{R}^d$ denote the union of n distinct lines through the origin. Compute $\pi_1(\mathbb{R}^d \setminus X)$.

(H2) Consider the space X obtained by identifying opposite sides of the cube I^3. Find a cell decomposition of X, and use this to find its fundamental group.

(H3) Consider the space X obtained by identifying opposite sides of the cube I^3, each with a "quarter twist" clockwise (the particular rotation direction will not affect the final space). Find a cell decomposition of X, and use this to find its fundamental group.

(H4) Show that if X and Y are path connected, then $\pi_1(X * Y)$ is trivial.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: every finitely generated group is the fundamental group of some cell complex with finitely many cells.