Spring 2021, Math 621: Problem Set 7
Due: Thursday, March 18th, 2021
Covering Spaces

(D1) *Universal covering spaces.* A covering space $\widetilde{X}$ of a path connected space X is said to be a *universal covering space* if $\widetilde{X}$ is path connected and simply connected ($\pi_1(\widetilde{X})$ is trivial).

Find a universal covering space for each of the following spaces.

(a) The circle S^1.
(b) The sphere S^2.
(c) The torus $S^1 \times S^1$.
(d) Real projective space $\mathbb{RP}^d$.
(e) The wedge product $S^2 \vee S^1$.
(f) A bouquet of 2 circles $S^1 \vee S^1$.

(D2) *Fundamental groups of covering spaces.* For each subgroup $H \subset G$, identify a space (X, x_0) and a covering space $(\widetilde{X}, \widetilde{x}_0)$ so that $\pi_(X, x_0) \cong G$ and $\pi_1(\widetilde{X}, \widetilde{x}_0)$ corresponds to H.

Note: unless otherwise stated, the subgroups here are generated **as subgroups**, not as normal subgroups.

(a) $H = (2\mathbb{Z}) \times \mathbb{Z} \subset G = \mathbb{Z} \times \mathbb{Z}$.
(b) $H = \{0\} \times \mathbb{Z} \subset G = \mathbb{Z} \times \mathbb{Z}$.
(c) $H = \langle a \rangle \subset G = \langle a, b \rangle$.
(d) $H = \langle bab^{-1}, b^2ab^{-2}, \ldots \rangle \subset G = \langle a, b \rangle$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find a universal covering space of the union of S^2 and a diameter.

(H2) Find a universal covering space of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.

(H3) Let $T = S^1 \times S^1$. Locate a covering space map $p : T \to K$, where K is the Klein bottle. Identify the corresponding subgroup of $\pi_1(K)$ corresponding to $\pi_1(T)$.

(H4) Find a covering space of $S^1 \vee S^1$ corresponding to the normal subgroup $H = \langle a^2, b^2, (ab)^4 \rangle$ of $\pi_1(S^1 \vee S^1) \cong \langle a, b \rangle$.

(H5) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) Given universal covering space $\widetilde{X}$ and $\widetilde{Y}$ for spaces X and Y respectively, the space $\widetilde{X} \times \widetilde{Y}$ is a universal covering space of $X \times Y$.

(b) Given universal covering space $\widetilde{X}$ and $\widetilde{Y}$ for spaces X and Y respectively, the space $\widetilde{X} \vee \widetilde{Y}$ is a universal covering space of $X \vee Y$.