Spring 2021, Math 621: Problem Set 8
Due: Thursday, March 25th, 2021
The Fundamental Theorem of Covering Spaces

(D1) *The full Galois correspondence.* Find all path-connected covering spaces of the given space.

(a) $X = \mathbb{RP}^d$

(b) $X = \mathbb{RP}^2 \times \mathbb{RP}^2$

(c) X is the space obtained from S^1 by attaching a single 2-cell that wraps around 4 times.

(D2) *Deck transformations and monodromy.* Given a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, the group $\pi_1(X, x_0)$ acts on the fiber $p^{-1}(x_0)$ via path lifting in the following way: given $x \in p^{-1}(x_0)$ and a loop $\gamma : I \to X$ based at x_0, define $[\gamma] \cdot x = y$ when γ lifts to a path $\widetilde{\gamma} : I \to \widetilde{X}$ with $\widetilde{\gamma}(0) = x$ and $\widetilde{\gamma}(1) = y$. This is known as the *monodromy action.*

(a) Draw a general illustration of this action. Be sure to label x_0, x, and y.

(b) Suppose $p : (\widetilde{X}, \widetilde{x}_0) \to (S^1, x_0)$ is a (not necessarily path-connected) 3-sheeted cover of S^1. Writing $p^{-1}(x_0) = \{x_1, x_2, x_3\}$, the monodromy action of each element of $\pi_1(S^1) \cong \mathbb{Z}$ naturally corresponds to an element of the permutation group S_3, and the set of such permutations is a subgroup of S_3. Determine which subgroups of S_3 are possible, and for each, locate a covering space $\widetilde{X}$.

(c) Suppose $\widetilde{X}$ is the universal cover of X. In lecture, we obtained a natural isomorphism $\mathrm{Gal}(\widetilde{X}, X) \cong \pi_1(X, x_0)$. This defines an action of $\pi_1(X, x_0)$ on $p^{-1}(x)$ obtained by restricting each deck transformation in $\mathrm{Gal}(\widetilde{X}, X)$ to the set $p^{-1}(x_0)$.

(i) Verify this action does **not** coincide with the monodromy action for $X = S^1 \vee S^1$. Hint: write $\pi_1(S^1 \vee S^1, x_0) = \langle a, b\rangle$ in the usual way, and let $\widetilde{x}_0$ and $\widetilde{x}_1$ denote the endpoints of the path obtained by lifting the loop b to $\widetilde{X}$. Where does each action by a send the vertices $\widetilde{x}_0$ and $\widetilde{x}_1$?

(ii) Verify this action **does** coincide with the monodromy action when $X = S^1 \times S^1$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Locate a covering space $\widetilde{X}$ of the torus $T = S^1 \times S^1$ whose corresponding subgroup of $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$ is $H = \langle (1,2), (3,2) \rangle$.

(H2) Let K denote the Klein bottle. Construct a non-normal covering space map $K \to K$.

(H3) Construct finite graphs X_1 and X_2 and a finite graph $\widetilde{X}$ that is a covering space of both X_1 and X_2, but so that there is no space having both X_1 and X_2 as covering spaces.

(H4) Find all path-connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.