Spring 2021, Math 621: Problem Set 9
Due: Thursday, April 8th, 2021
Venturing into Homological Algebra

(D1) *Short exact sequences.* A chain complex of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence* if $\ker(f) = 0$, $\mathrm{Im}(f) = \ker(g)$, and $\mathrm{Im}(g) = C$ (i.e., if the sequence is *exact* everywhere).

(a) Define maps f and g so that

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^3 \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}_3 \longrightarrow 0$$

is exact.

(b) Consider the short exact sequence of Abelian groups at the start of this problem.

(a) Prove exactness holds at A if and only if f is injective.

(b) Prove exactness holds at C if and only if g is surjective.

(c) Suppose A, B, C are Abelian groups with $A \subset B$, and suppose

$$0 \longrightarrow A \hookrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence. Express C in terms of A and B.

(D2) *Split exact sequences.* A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is said to *split* if there exists an isomorphism $h : A \oplus C \longrightarrow B$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \hookrightarrow & A \oplus C & \twoheadrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

commutes (where the maps in the top row are the "natural" maps to/from $A \oplus C$).

(a) Prove that the short exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^3 \xrightarrow{g} \mathbb{Z} \longrightarrow 0$$

splits, where

$$f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad g = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}.$$

(b) Prove that any short exact sequence of vector spaces over $\mathbb{Q}$ splits.

(c) Does part (b) hold for finite dimensional vector spaces over any field $\Bbbk$?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Fix a finite set $V = \{0, 1, 2, \ldots, n\}$, and let $\Delta = 2^V$ denote the power set of V. For each integer $d = -1, 0, 1, \ldots, n$, define the Abelian group

$$C_d = \bigoplus_{\substack{F \in \Delta \\ |F| = d+1}} \mathbb{Z}$$

and let e_F denote the basis vector of C_d correspoding to $F \in \Delta$. Define the group homomorphism $\partial_d : C_d \to C_{d-1}$ by defining

$$\partial_d(e_F) = \sum_{j=0}^{d} (-1)^j e_{F \setminus \{i_j\}}$$

for each basis vector $e_F \in C_d$, where $F = \{i_0 < \cdots < i_d\}$. Prove that

$$0 \longrightarrow C_n \xrightarrow{\partial_d} \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is a complex.

(H2) Given any B and a complex $C_\bullet$ of the form

$$0 \longrightarrow C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \longrightarrow \cdots$$

we can use the covariant functor $\mathrm{Hom}(B, -)$ to obtain a complex

$$0 \longrightarrow \mathrm{Hom}(B, C_0) \xrightarrow{f_0^*} \mathrm{Hom}(B, C_1) \xrightarrow{f_1^*} \mathrm{Hom}(B, C_2) \longrightarrow \cdots$$

which we will denote $\mathrm{Hom}(B, C_\bullet)$.

(a) Prove that $\mathrm{Hom}(B, C_\bullet)$ is indeed a complex.

(b) Given an arbitrary B, use the contravariant functor $\mathrm{Hom}(-, B)$ to define a complex $\mathrm{Hom}(C_\bullet, B)$ obtained from any complex $C_\bullet$. You do **not** need to prove that $\mathrm{Hom}(C_\bullet, B)$ is indeed a complex!

(H3) Suppose $C_\bullet$ is an exact sequence

$$0 \longrightarrow C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_d \longrightarrow 0$$

of finite dimensional vector spaces over a field $\Bbbk$. Prove that

$$\sum_{i=0}^{d} (-1)^i \dim_{\Bbbk} C_i = 0.$$

(H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) Any short exact sequence of finitely generated Abelian groups splits.

(b) If the complex $C_\bullet$ in (H2) is exact, then so is $\mathrm{Hom}(B, C_\bullet)$.