Spring 2021, Math 621: Problem Set 11
Due: Thursday, April 22nd, 2021
Simplicial and Singular Homology

(D1) *Free objects.* An Abelian group C is *free* if $C \cong \bigoplus_{j \in J} \mathbb{Z}$ for some index set J. Analogously, a vector space V over a field $\Bbbk$ is called *free* if $K \cong \bigoplus_{j \in J} \Bbbk$ for some index set J.

(a) Explain why every vector space is free. Demonstrate that the same is **not** true for Abelian groups.

(b) Draw a "universal property" diagram for the following statement about an object P.

> Given any surjection $f : A \to B \to 0$ and any homomorhism $g : P \to B$, there exists a homomorphism $h : P \to A$ such that $f \circ h = g$.

(c) Prove the statement in the previous part if P is a free Abelian group.

(d) Given a nearly identical proof if P is a (free) vector space.

(e) Locate an Abelian group P that does **not** satisfy the statement in part (b).

(D2) *Induced homomorphisms on homology.* Suppose $f : C_\bullet \to D_\bullet$ is a morphism of complexes.

(a) Define a homomorphism $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ induced by f, and, using diagram chasing, prove that your homomorphism is well-defined.

(b) Prove that if f is an isomorphism of complexes (that is, if each $f_i : C_i \to D_i$ is an isomorphism), then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism as well.

(c) Suppose $X = A \cup B$. Use parts (a) and (b) to show $H_n(X;\mathbb{Z}) \cong H_n(A;\mathbb{Z}) \oplus H_n(B;\mathbb{Z})$.

(d) Suppose $g : X \to Y$ is a continuous map. Use g to define a morphism of complexes $C_\bullet(X;\mathbb{Z}) \to C_\bullet(Y;\mathbb{Z})$, and prove that f is indeed a morphism of complexes.

(e) Conclude $H_n(-;\mathbb{Z})$ is a functor Top $\to$ Ab. Is it covariant or contravariant?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find a triangulation of $S^1 \vee S^2$, and compute the simplicial homology over $\mathbb{Q}$.

(H2) Consider the annulus $X \subset \mathbb{R}^2$ centered at the origin with inner radius 1 and outer radius 2. Let $\sigma_1, \sigma_2 : \Delta^1 \to X$ denote the half-circle arcs along the top and bottom of the inner circle, respectively, each with endpoints on the x-axis and traversed counter-clockwise. Likewise, let $\tau_1, \ldots, \tau_4 : \Delta^1 \to X$ denote the 4 quarter-arcs along the outer circle, with each τ_i in the i-th quadrant and both endpoints on axes, again traversed counter-clockwise.

Let $c = e_{\sigma_1} + e_{\sigma_2}$ and $c' = e_{\tau_1} + e_{\tau_2} + e_{\tau_3} + e_{\tau_4}$. Prove $c, c' \in \ker \partial_1$ and $[c] = [c'] \in H_1(X; \mathbb{Z})$.

(H3) (a) Prove the category of complexes (of vector spaces or Abelian groups) has direct sums.

(b) Fix a category $\mathcal{C}$ having direct sums and a 0 object. An object C in $\mathcal{C}$ is *decomposable* if it is isomorphic to a direct sum of nonzero objects in $\mathcal{C}$, and *indecomposable* otherwise. Determine which objects are indecomposable in the category of complexes of vector spaces over $\mathbb{Q}$.

(H4) Fix B and a complex $C_\bullet$ of the form

$$0 \longrightarrow C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \longrightarrow \cdots .$$

Prove that if $C_\bullet$ is exact and each C_i is free, then $\operatorname{Hom}(B, C_\bullet)$ is exact.

(H5) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If P is free, then the homomorphism h in Problem (D1) is unique.

(b) If $f : C_\bullet \to D_\bullet$ induces an isomorphism on homology, then f is an isomorphism of chain complexes.