**Spring 2021, Math 621: Problem Set 13**
**Due: Thursday, May 6th, 2021**
**The Long Exact Sequence of Homology**

(D1) *Using the long exact sequence of homology.*

(a) Let $X = S^1 \amalg S^1$, and let $A \subset X$ denote a 2-element set with one point on each circle. Use the long exact sequence of the good pair $(X, A)$ to find $\widetilde{H}_0(S^1 \vee S^1; \mathbb{Z})$ and $\widetilde{H}_1(S^1 \vee S^1; \mathbb{Z})$.

(b) More generally, express $\widetilde{H}_n(X \vee Y; \mathbb{Z})$ in terms of $\widetilde{H}_n(X; \mathbb{Z})$ and $\widetilde{H}_n(Y; \mathbb{Z})$.

(D2) *The snake lemma.* The goal of this problem is to prove that, given a short exact sequence

$$0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0$$

of complexes, there is a long exact sequence

$$\cdots \longrightarrow H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \longrightarrow \cdots$$

where the map $\partial$ is as defined in the lecture preceeding this problem.

(a) Prove that $g_* \circ f_* = 0$, ensuring that $\operatorname{Im} f_* \subseteq \ker g_*$.

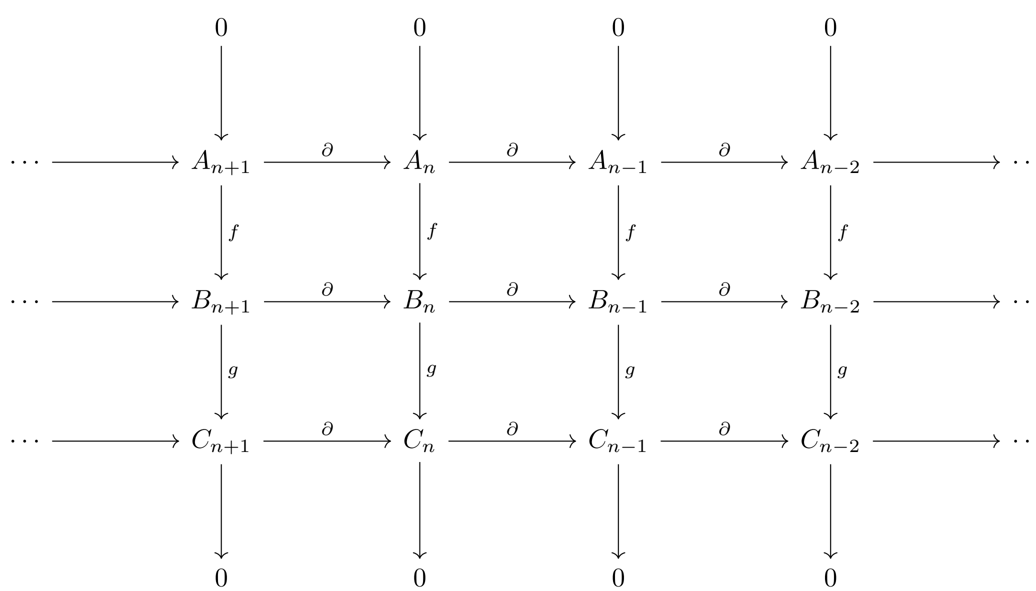
(b) Prove that $\partial \circ g_* = 0$, ensuring that $\operatorname{Im} g_* \subseteq \ker \partial$.

(c) Prove that $f_* \circ \partial = 0$, ensuring that $\operatorname{Im} \partial \subseteq \ker f_*$.

(d) Prove that $\operatorname{Im} f_* \supseteq \ker g_*$.

(e) Prove that $\operatorname{Im} g_* \supseteq \ker \partial$.

(f) Prove that $\operatorname{Im} \partial \supseteq \ker f_*$.



**Homework problems.** You must submit *all* homework problems in order to receive full credit.

(H1) (a) Let $X = S^1 \times S^1$ denote the torus, presented as a square with sides identified in the usual way, and let $A = S^1 \vee S^1 \subset X$ denote the border of the square. Use the long exact sequence of the good pair $(X, A)$ to find $\widetilde{H}_1(X;\mathbb{Z})$ and $\widetilde{H}_2(X;\mathbb{Z})$.

(b) Let $X$ denote the Klein bottle, presented as a square with sides identified in the usual way, and let $A = S^1 \vee S^1 \subset X$ denote the border of the square. Use the long exact sequence of the good pair $(X, A)$ to find $\widetilde{H}_1(X;\mathbb{Z})$ and $\widetilde{H}_2(X;\mathbb{Z})$.

(H2) Find the relative homology groups $H_n(X, A;\mathbb{Z})$ where $X = S^1 \times S^1$ and $A$ is any finite subset of $X$ containing $k$ points.

(H3) Suppose $X$ is a $d$-dimensional cell complex (that is, all cells have dimension at most $d$).

(a) Prove that $H_n(X;\mathbb{Z}) = 0$ for $n > d$.

(b) Prove that $H_d(X;\mathbb{Z})$ is free.