Spring 2021, Math 621: Project Topics

The goal of each project is to learn about a topic not discussed in class. Throughout the semester, the following will be expected.

- Choose a topic. Please speak with me before making your decision, to ensure it is an appropriate level and so that we can narrow down a reasonable set of goals. You should choose a topic (and have it approved) no later than **Monday, April 5th**.
- Begin reading the agreed-upon background material. Plan to **meet at least twice** with me throughout the rest of the semester, to ensure that you are on track.
- Write (in LaTeX) a paper aimed at introducing your topic to fellow students, containing ample examples and explanations in addition to any theorems and proofs you give. Your writing should convey that you understand the intricacies of any proofs presented. Keep the following deadlines in mind as you proceed.
  - A rough draft of the paper will be due **Friday, April 30th** (one week before the last day of class). This will be peer reviewed by a fellow student in the following week.
  - The final paper will be due on **Thursday, May 13th** (our "final exam" day).
- Give a 10-15 minute presentation introducing the main ideas of your chosen project topic. Presentations will take place (virtually) during the final exam slot at the semester's end. You should keep in mind your target audience and time constraints when deciding what and how to present.
- Your final grade on the project will be determined by the content, quality, and completeness of your final writeup, and on the quality of your presentation.

Given below are several project ideas. Many of the listed sources contain more material than is necessary for the project, so be sure to meet with me so we can set reasonable project goals. I am also open to projects not listed here, but you must run them by me before making a decision. Don't be afraid to ask questions at any point during the project!

General topology

(1) *Exotic examples of topological spaces.* A large portion of point-set topology involves classification and properties of general topological spaces. This area is notorious for having a wealth of "pathological" examples that illustrate various strange and subtle behaviors.

Source: *Counterexamples in Topology* (Steen and Seebach)

(2) *Poincaré conjecture.* Considered one of the first open problems in topology, the Poincaré conjecture states that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. It remained open for around 100 years, and was finally solved in the early 2000's.

Source: *The Poincaré Conjecture: In Search of the Shape of the Universe* (O'Shea)

Note: This source is more about the history of the conjecture, its eventual solution, and the mathematicians involved, than it is about the mathematics of the conjecture itself. Although it is intended for a "lay-audience" it is incredibly well-written.

Homotopy theory

(3) *Higher homotopy groups.* The groups $\pi_n(T)$, known as higher homotopy groups, are analogs of the fundamental group $\pi_1(T)$ for higher dimensional holes. Stated in terms of analogies, higher homotopy groups are to the fundamental group as the groups $H_n(\mathbb{Z})$ for $n \geq 2$ are to $H_1(\mathbb{Z})$. Unlike homology groups, the higher homotopy groups are in general very difficult to pin down exactly (even for spheres, they are not all known).

Source: *Algebraic Topology* (Hatcher), Chapter 4

(4) *Mapping class groups and braid groups.* Braid groups are constructed from concatenating tangled strands of string. The can be modeled and studied topologically using configuration spaces, which track the relative positions of points in a space, and mapping class groups, which in general encode the symmetries of a topological space T via the homeomorphisms $T \to T$ (the automorphisms of T) modulo isotopy equivalence.

Source: *Braids: A Survey* (Birman, Brendle)

(5) *Knots, links, and quandles.* Knot theory is the study of, well, knots: string tangled up in space. One of the biggest questions is how to determine if two knots are "equivalent", i.e., if one can be (un)tangled into the other without cutting the string. Quandles are algebraic objects derived from the topology of (the set-complement of) a knot.

Source: *Quandles: An Introduction to the Algeba of Knots* (Elhamdadi, Nelson)

Homology

(6) *Persistence homology.* Unlike "regular" homology, which is obtained from a fixed topological space, persistence homology tracks the "birth" and "death" (yes, these are technical terms) of cycles as a space changes with time. For example, one can think of a collection of disks in the plane with radii slowly growing, so that over time, the number of connected components will drop as the disks being to intersect, and some 1-dimensional holes may form and then eventually disappear when they are filled in.

Source: *Persistence Homology - A Survey* (Edelsbrunner, Harer)

(7) *Alexander duality.* If a 2-sphere S^2 is partitioned into 2 sufficiently nice subspaces X and Y, then, roughly speaking, the number of connected components of X should equal the number of 1-dimensional holes of Y, and visa-versa. Alexander duality is a higher dimensional analog of this idea, relating the homology of two subspaces X and Y that partition S^n.

Source: *Combinatorial Alexander Duality - A Short and Elementary Proof* (Björner, Tancer)

(8) *Cohomology.* The categorical dual of "regular" homology. Although it is obtained purely algebraically from the same chain complex used to obtain homology, cohomology somehow extracts extra topological data about the original space.

Source: *Algebraic Topology* (Hatcher), Chapter 3

(9) *Morse theory.* Morse theory studies sufficiently nice "height functions" $T \to \mathbb{R}$ on a topological space T. Discrete Morse theory, a more recent development, adapts Morse theory for working (combinatorially) with cell complexes.

Source: *A User's Guide to Discrete Morse Theory* (Forman)

(10) *Computational topology.* When we encounter a new area of study (e.g., algebraic topology), one of the natural first questions is “how can we make a computer work out examples for us?” It turns out simplicial complexes are a natural data structure for encoding topological spaces, and there are algorithms for programmatically constructing simplicial chain complexes (and hence homology).

Source: *Computational Topology* (Edelsbrunner, Harer)

Note: this project requires some familiarity with programming.

Combinatorics

(11) *Topological combinatorics.* Simplicial complexes arise naturally in numerous combinatorial and algebraic settings, and their topological properties (e.g., homology) often have combinatorial interpretations. The h-vector, which can be expressed in terms of the number of faces of each dimension, has both topological and combinatorial interpretation when the simplicial complex is shellable.

Sources: *Lecture Notees on Algebraic Combinatorics* (Martin)
Poset Topology: Tools and Applications (Wachs), Lecture 3

(12) *Poset homology.* Posets are fundamental combinatorial objects that arise naturally in countless areas of mathematics. Each poset has an associated simplicial complex, called the order complex, whose topology encodes structural information about the poset.

Source: *Poset Topology: Tools and Applications* (Wachs), Lecture 1

Applications

(13) *Topological data analysis.* Homological tools from topology can be used in clever ways when analyzing large sets of data. Such methods have a multitude of applications across the scientific landscape.

Source: *An Introduction to Topological Data Analysis: Fundamental and Practical Aspects for Data Scientists* (Chazal, Michel)

(14) *Compressive sensing.* From the area of signal processing, compressive sensing uses a variety of methods (usually analytical in nature) to recover a signal from incomplete data. Recently, tools from algebraic topology have been used to this end as well.

Source: *Persistent Homological Structures in Compressed Sensing and Sparse Likelihood* (Chung, Lee, Arnold).

Note: this source is somewhat terse, so additional sources on background may be needed.