Spring 2022, Math 579: Week 3 Problem Set
Due: Thursday, February 17th, 2022
Binomial Theorem and Combinatorial Proofs

Discussion problems. The problems below should be worked on in class.

(D1) *Combinatorial proofs.*

(a) Compare your answers to the preliminary problems, and come to a concensus.

(b) Fill in the blanks in the following **combinatorial** proof that for any $n \geq 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Proof. We will count the number of ways to choose subsets $A, B \subseteq [n]$ with $A \subseteq B$.

Right side: for each $i \in [n]$, either $i \in A$, $i \in B \setminus A$, or ____. This yields 3^n possibilities.

Left side: if we let $k = |B|$, then for each $k = 0, 1, \ldots, n$, there are ____ choices for B, and ____ ways to choose a subset $A \subseteq B$. This yields

$$2^0\binom{n}{0} + 2^1\binom{n}{1} + \cdots + 2^n\binom{n}{n}$$

possibilities. We conclude the left side must equal the right side. □

(c) In the proof of the previous part, replace "$A, B \subseteq [n]$ with $A \subseteq B$" in the first line with "$C, D \subseteq [n]$ with $C \cap D = \emptyset$", and then rewrite the rest of the proof accordingly.

(d) Give a combinatorial proof that whenever $0 \leq k \leq n - 3$,

$$\binom{n}{3}\binom{n-3}{k} = \binom{n}{k}\binom{n-k}{3}.$$

(e) Give a combinatorial proof of the following identity for $n \geq 1$:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

(D2) *The binomial theorem.* Recall the binomial theorem from Thursday:

$$(x+z)^n = \sum_{k=0}^{n} \binom{n}{k} x^k z^{n-k}.$$

(a) Use the binomial theorem to find the coefficient of $x^9 z^{15}$ in the expression $x^5(x^2 - z)^{17}$.

(b) Use the binomial theorem to prove that for any $n \geq 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

We call this an **algebraic** proof.

(c) Apply the binomial theorem (3 times!) to the expression

$$(x+1)^{n+1} = x(x+1)^n + (x+1)^n.$$

Then, reindex each sum to contain x^{k+1} (as opposed to x^k) and pull out terms so that each sum starts at $k = 1$ and ends at $k = n$. Lastly, consolidate the right hand side into a single sum. Comparing coefficients on the left and right hand sides, what identity is obtained?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the coefficient of $x^{11}z^7$ in the expansion of $(x+z)^{18} + x^3(x-z)^{15}$.

Hint: **don't expand**!!! This is what the binomial theorem is for!!!

(H2) Use induction on n to prove that for all $n \geq 1$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: use the identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

in your inductive step.

(H3) Give a **combinatorial** proof that for all $n \geq 1$,

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = 2^{n-2}n(n-1).$$

(H4) Give an **algebraic** proof that for $n \geq 1$,

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}.$$

Hint: write 3 and -1 each in a clever way, and then use the binomial theorem twice.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give an **algebraic** proof that for any $n, m \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, we have

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{m+j}{m} = \begin{cases} (-1)^n & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}$$