

Spring 2022, Math 579: Week 5 Problem Set
Due: Thursday, March 3rd, 2022
Integer Partitions

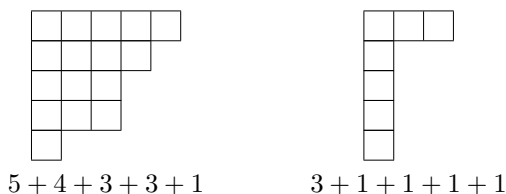
Discussion problems. The problems below should be worked on in class.

(D1) *Compositions vs. integer partitions.*

- (a) Compare your answers to the prelim problems.
- (b) Find the number of (strong) compositions of 10 into all even parts. Compare to (P1).
- (c) Based on your observations in the previous part, state and prove a conjecture about the number of (strong) compositions of n into even parts.
- (d) Show $p(6) = 11$ and $p_3(7) = 4$ by listing the corresponding partitions.
- (e) Give a combinatorial proof that for $n \geq 2$, the number of partitions of n with no 1's is given by $p(n) - p(n - 1)$.

(D2) *Ferres diagrams.* Recall that $p(n)$ counts the number of integer partitions of n , and $p_k(n)$ counts the number of partitions into exactly k parts.

- (a) Given a partition of n , the associated *Ferrer diagram* or *Young diagram* is a drawing of n boxes that are “upper-left justified” where the number of boxes in the i 'th row equals the size of the i 'th part. See below for examples.



Find all partitions of 7 into 3 parts, and write the Ferrer diagram for each partition.

- (b) Given a Ferrer diagram of a partition, what is the number of rows equal to? What is the number of columns equal to?
- (c) Using the previous part, find a bijection between the partitions of n into at most k parts and the partitions of n into parts of size at most k .
- (d) Based on your bijection in the previous part, is it true that $p_k(n)$ equals the number of partitions of n into parts of size exactly k ?
- (e) The partition obtained by reflecting a Ferrer diagram about its diagonal (which you likely used in the preceding few problems) is called the *conjugate* partition. Explain why the conjugate of a partition in which every part appears an even number of times is a partition whose parts are all even. Is the converse true?
- (f) Describe the partitions whose conjugate partition has no parts of size 1.
- (g) Prove that the number of partitions of n into at most k parts equals $p_k(n + k)$.
- (h) A partition is *self-conjugate* if it equals its conjugate. Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

Homework problems. You must submit *all* homework problems in order to receive full credit.

- (H1) Prove that for $n \geq 2$, the number of partitions of n in which the two largest parts are equal is given by $p(n) - p(n - 1)$.
- (H2) A partition is *self-conjugate* if it equals its conjugate. Prove the number of self-conjugate partitions of n has the same parity as $p(n)$ (that is, the number of self-conjugate partitions of n is even if and only if $p(n)$ is even).
- (H3) Find an expression for the number of partitions of n in which each part appears an even number of times (your expression is allowed to use $p(k)$ for $k \leq n$). For example, this includes the partition $4 + 4 + 3 + 3 + 3 + 3 + 1 + 1$, but not $8 + 3 + 3 + 1 + 1$.
- (H4) Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.
- (H5) Let a_n denote the number of strong compositions of n into parts that are larger than 1. For $n \geq 4$, find a formula for a_n in terms of a_{n-1} and a_{n-2} .

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

- (C1) Prove that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.