Spring 2022, Math 579: Week 7 Problem Set
Due: Thursday, March 17th, 2022
Introduction to Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Formal power series.*

(a) Fill in the blank in each of the following.

(i) $\displaystyle\sum_{n=0}^{\infty} \_\_\_\_ z^{\_\_} = \frac{2}{1-3z^2}$ (ii) $\displaystyle\sum_{n=0}^{\infty} \_\_\_\_ z^{\_\_} = \frac{z}{2+z}$

(b) Write each of the following as a rational function in z.

(i) $\displaystyle A(z) = \sum_{n=0}^{\infty} 2^{n-1} z^n$ (ii) $\displaystyle A(z) = \sum_{n=0}^{\infty} (-1)^n 2^{2n} z^n$

(c) Find a formula for a_n in terms of n given the following $A(z)$.

(i) $\displaystyle A(z) = \frac{3z^2+2z+17}{1-z^2}$ (ii) $\displaystyle A(z) = \frac{3z^2+2z+17}{(1-z)^2}$

(d) For each of the following $A(z)$, locate a formula for the coefficients of $B(z) = 1/A(z)$. Then, prove via direct multiplication that $A(z)B(z) = 1$.

(i) $A(z) = 1 - 3z + 2z^2$
Hint: find the first few coefficients of $B(z)$ by hand using the fact $A(z)B(z) = 1$, then conjecture a formula for b_n, then prove your conjecture.

(ii) $A(z) = 1 - z + z^2 - z^3 + \cdots = \sum_{n\geq 0}(-1)^n z^n$
Hint: start by writing $A(z)$ as a rational function of z.

(D2) *Power series derivatives.* The "formal derivative" of $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is defined

$$A'(z) = \frac{d}{dz}A(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n.$$

It turns out, formal derivatives satisfy Calculus 1 derivative rules (product rule, etc.).

(a) Take the formal derivative of the geometric series to verify (now, a third way) that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n = 1 + 2z + 3z^2 + \cdots$$

(b) Manipulate the above expression to write $\sum_{n=0}^{\infty} nz^n$ as a rational expression in z.

(c) Use "formal differentiation" to express $\sum_{n=0}^{\infty} n^2 z^n$ as a rational expression in z.

(D3) *Consistency.* The goal of this problem is to do some "sanity checks" on formal power series derivative rules. Recall the power series

$$e^z = \sum_{n=0}^{\infty}\frac{1}{n!}z^n = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots.$$

(a) Verify the term-by-term derivative of the series for e^{2z} has the same coefficients as the series for $2e^{2z}$.

(b) Find the derivative of $z^2 e^z$ using (i) term-by-term differentation, and (ii) using standard derivative rules, and verify they are equal.

(c) Verify that

$$\frac{e^z - 1}{1-z} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k!}\right)z^n.$$

(d) Verify that the term-by-term derivative of the series in part (c) matches the one obtained from quotient rule.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use term-by-term derivatives to obtain a formula for a_n for each of the following power series $A(z) = \sum_{n\geq 0} a_n z^n$. You may use the series

$$\frac{1}{(1-z)^2} = \sum_{n\geq 0}(n+1)z^n \qquad \text{or} \qquad \frac{z}{(1-z)^2} = \sum_{n\geq 0} nz^n$$

from class as a starting place.

(a) $A(z) = \dfrac{z+z^2}{(1-z)^3}$ (b) $A(z) = \dfrac{z+4z^2+z^3}{(1-z)^4}$

(H2) For each of the following sequences, find an expression in terms of z for $A(z) = \sum_{\geq 0} a_n z^n$. Your answer should **not** contain any "$\cdots$" or sigma-sums.

Hint: you can substantially shorten the algebra needed to obtain your answer with clever use of power series we have already encountered.

(a) $a_n = n^3 + 7n^2 + 3n - 5$ (b) $a_n = n2^n - n^2 3^n + \dfrac{1}{(n+2)!}$

(H3) (a) Prove that $A(z) = B(z)$ if and only if $a_0 = b_0$ and $A'(z) = B'(z)$.

(b) Consider the power series

$$S(z) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{3!}z^3 + \cdots \qquad \text{and} \qquad C(z) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{1}{2!}z^2 + \cdots$$

Argue that $(S(z))^2 + (C(z))^2 = 1$. Thinking back to Calculus 2, what familiar identity does this equality encode?

Hint: this can be done without heavy power series algebra! First, show $S'(z) = C(z)$ and $C'(z) = -S(z)$, and use this to show

$$2S(z)S'(z) + 2C(z)C'(z) = 0.$$

Then use part (a).

(H4) Define $c(n) = 1$ if there exists a way to write n as a sum of the values 3 and 5, and $c(n) = 0$ otherwise. For instance, $c(13) = 1$ since $13 = 3 + 5 + 5$, and $c(15) = 1$ since $15 = 5 + 5 + 5$ (as well as $15 = 3 + 3 + 3 + 3 + 3$), but $c(7) = 0$ since there is no way to add 3's and 5's together to obtain 7. Prove that

$$\sum_{n=0}^{\infty} c(n) z^n = \frac{1 - z^{15}}{(1-z^3)(1-z^5)}.$$

Note: $c(0) = 1$ since $0 = 0 \cdot 3 + 0 \cdot 5$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Verify the "formal derivative rules": if $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$, then

(i) $\dfrac{d}{dz}\big(A(z)B(z)\big) = A'(z)B(z) + A(z)B'(z)$ (the product rule);

(ii) $\dfrac{d}{dz}\left(\dfrac{A(z)}{B(z)}\right) = \dfrac{A'(z)B(z) - A(z)B'(z)}{(B(z))^2}$ (the quotient rule); and

(iii) $\dfrac{d}{dz}\big(A(B(z))\big) = A'(B(z))B'(z)$ (the chain rule).

Be sure to indicate any necessary assumptions on $B(z)$ in parts (ii) and (iii)!