Spring 2022, Math 579: Week 8 Problem Set
Due: Thursday, March 24th, 2022
Recurrence Relations

Discussion problems. The problems below should be worked on in class.

(D1) *Solving recurrence relations with generating functions.* For each of the following, (i) compute $a_0, \ldots, a_5$, (ii) use generating functions to find a formula for a_n in terms of n, and (iii) verify your formula for $n \le 5$.

(a) $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-2}$ for $n \ge 2$.
(b) $a_0 = 1$, $a_n = 3a_{n-1} - 2$ for $n \ge 1$.
(c) $a_0 = 3$, $a_1 = 1$, $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \ge 2$.
(d) $a_0 = 1$, $a_n = 2a_{n-1} + n$ for $n \ge 1$.
(e) $a_0 = 3$, $a_n = na_{n-1} - n^2$ for $n \ge 1$.
Hint: use exponential generating functions for this one!

(D2) *Generating functions and integer partitions.* Consider the identity

$$\sum_{n\ge0} p(n)z^n = \left(\frac{1}{1-z}\right)\left(\frac{1}{1-z^2}\right)\left(\frac{1}{1-z^3}\right)\cdots = \prod_{k\ge1}\frac{1}{1-z^k},$$

where $p(n)$ denotes the number of integer partitions of n.

(a) First, let's see why the above identity holds.
(i) Verify that the coefficient of z^5 is 7 on **both sides** of the equation.
Hint: on the right hand side, expand each parenthetical using the geometric series.
(ii) Justify why the coefficient of z^n on the right hand side equals $p(n)$.

(b) Consider the power series

$$\sum_{n\ge0} a_n = \frac{1}{(1-z^2)(1-z^3)}.$$

(i) Complete the interpretation of a_n: "# of integer partitions of n into ________".
(ii) Use partial fractions to find a formula for a_n.
Hint: this can be done without resorting to radicals or complex roots! Use the factorizations $1-z^2 = (1-z)(1+z)$ and $1-z^3 = (1-z)(1+z+z^2)$.

(c) Let $p_{\text{even}}(n)$, $p_{\text{odd}}(n)$, and $p_{\text{dist}}(n)$ denote the number of integer partitions of n into only even parts, into only odd parts, and into distinct parts, respectively.
(i) Adjust the above identity to obtain analogous "infinite product" expressions for

$$\sum_{n\ge0} p_{\text{even}}(n)z^n, \qquad \sum_{n\ge0} p_{\text{odd}}(n)z^n, \qquad \text{and} \qquad \sum_{n\ge0} p_{\text{dist}}(n)z^n.$$

Hint: the third series will **not** require denominators.
(ii) Using your expressions, give a generating functions proof that $p_{\text{odd}}(n) = p_{\text{dist}}(n)$.
Hint: do not expand to coefficients! Write out the first 5 parentheticals in each product, and look for ways to cleverly factor and cancel.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use generating functions to find a_n if $a_0 = 1$ and $a_n = 3a_{n-1} + 2^n$ for $n \geq 1$.

(H2) Use generating functions to find F_n if $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
Hint: verify first that $\omega + \overline{\omega} = 1$, $\omega \cdot \overline{\omega} = -1$, and $1 - z - z^2 = (1 - \omega z)(1 - \overline{\omega} z)$, where

$$\omega = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \overline{\omega} = \frac{1-\sqrt{5}}{2}.$$

(H3) Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 1$, and $a_n = na_{n-1} + n(n-1)a_{n-2}$ for $n \geq 2$.
Hint: use exponential generating functions.

(H4) Consider the power series

$$\sum_{n \geq 0} a_n z^n = \frac{1}{(1-z^2)(1-z^3)}.$$

The goal of this problem is to find a formula for a_n in 2 different ways.

Hint: your two formulas should be identical, and will likely have 6 cases, based on the equivalence class of n modulo 6.

(a) Find a formula for a_n using partial fractions.
Hint: $1 - z^2 = (1-z)(1+z)$ and $1 - z^3 = (1-z)(1+z+z^2)$.

(b) Find a formula for a_n by first cleverly rewriting the rational expression so that the denominator is $(1-z^6)^2$.

(H5) Give a proof by induction that for each $d \geq 0$,

$$\sum_{n=0}^{\infty} n^d z^n = \frac{Q_d(z)}{(1-z)^{d+1}}$$

for some power series $Q_d(z)$ with finitely many nonzero terms.

Hint: start by identifying $Q_0(z)$, $Q_1(z)$, and $Q_2(z)$ using Problem (H1) from last week.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $d \geq 1$, and write the numerator in Problem (H5) as $Q_d(z) = a_d z^d + \cdots + a_1 z + a_0$. It turns out that $Q_d(1) = a_d + \cdots + a_1 + a_0 = d!$, which also equals the number of permutations of $[d]$. Locate a counting question involving permutations of $[d]$ to which a_m is the answer.