**Spring 2022, Math 579: Week 9 Problem Set**
**Due: Thursday, April 7th, 2022**
**Generating Functions for Combinatorics**

**Discussion problems.** The problems below should be worked on in class.

(D1) *Understanding this "A-structure" stuff.* Unless otherwise stated, let

$$A(z) = \sum_{n\geq 0} a_n z^n \qquad \text{and} \qquad B(z) = \sum_{n\geq 0} \frac{b_n}{n!} z^n$$

be the ordinary generating function of $a_n$ and exponential generating function of $b_n$, resp.

(a) Compare answers to the prelim problems within your group, and come to a concensus.

(b) First, let's consider a concrete example. Let $a_0 = 0$ and $a_n = 2^n$ for $n \geq 1$.

  (a) Let $c_n = n$ denote the number of ways to pick a person from a lineup of $n$ people, and let $C(z) = \sum_{n=1}^{\infty} c_n z^n$ denote its generating function. Using the composition formula, describe what the coefficients of $A(C(z))$ represent.

  (b) With $C(z)$ as in the previous part, interpret the coefficients of $C(A(z))$.

(c) Write down a formula for the coefficient of $z^n$ in the series $zA'(z)$ in terms of $a_n$. Give a combinatorial interpretation of this value that uses the word "$A$-structures".

(d) Does your answer to the previous part account for the case $n = 0$?

(e) Use the product formula to find a combinatorial interpretation of the coefficients of $zB(z)$ (as an exponential generating function). Use "$B$-structures" in your answer. Then, invent a counting question to which the coefficient of $z^n$ in $zB(z)$ is the answer.

(f) Suppose $C(z) = z/(1 - z)$. Interpret the coefficients of $B(C(z))$ (as an exponential generating function).

(D2) *Using generating functions to solve counting problems.*

(a) Use the product formula to find an expression for the ordinary generating function of the number $c_n$ of ways to take lineup of $n$ PE students, split the lineup into 2 (contiguous) teams (call them "left" team and "right" team), and choose a team leader from within the "left" team. Then, find a formula for $c_n$.

(b) What if the "right" team must have an even number of children in it? You do **not** have to find a formula this time, just an expression for the ordinary generating function.

(c) Use composition of ordinary generating functions to prove that the number of strong compositions of $n$ is $2^{n-1}$.

(d) Let $\ell_n$ denote the number of orderings of $[n]$, and let $L(z)$ denote its exponential generating function. Prove $L(z) = 1 + zL(z)$ combinatorially, then derive $\ell_n = n!$.

(e) Find the exponential generating function $T(z)$ for the number $t_n$ of ways to arrange $n$ books on two bookshelves so that each shelf has at least one book. Then, find a closed form for $t_n$.

(f) Suppose from our class of $n$ students, we select an odd number of students to serve on a committee, and select an even number of committee members to serve on a subcommittee. Find the exponential generating function for the number $c_n$ of ways to do this, and then use this to derive a closed formula for $c_n$.

(g) Suppose we have $n$ cards. We want to split them into an even number of nonempty subsets, form a line within each subset, then arrange the subsets in a line. Use generating functions to determine the number of different ways to do this.

(h) Prove $\sum_{n\geq 0} B(n) z^n = e^{e^z - 1}$, where $B(n)$ is the number of set partitions of $[n]$.

**Homework problems.** You must submit *all* homework problems in order to receive full credit.

(H1) For each of the following, find the an expression for the (ordinary) generating function $D(z) = \sum_{n=0}^{\infty} d_n z^n$ using the combinatorial interpretation of multiplication and composition of ordinary generating functions.

Note: your final answer to each part should be a rational expression for $D(z)$ in terms of $z$, and should not have any infinite sums or products. In particular, you are **not** required to find a formula for $d_n$!

(a) Suppose our course meets for $n$ days. Let $d_n$ denote the number of ways to split the available days into 3 units (counting methods, generating functions, and graph theory),

(i) select some (possibly empty) collection of days from the first unit for a pop quiz,

(ii) select an odd number of days from the second unit to hold discussions, and

(iii) select a single day from the third unit in which to give an exam.

(b) What if in part (a), each unit should be at least 2 days long?

(c) What if in part (a), exactly half of the days in the first unit must have a pop quiz, and they must alternate days (quiz, no quiz, quiz, no quiz, etc)?

(H2) Choose **one** of the parts of the previous problem, and use the generating function expression you obtained to find a formula for $d_n$. You may use a computer (e.g., `WolframAlpha`) to perform the partial fractions step for you.

(H3) Write solutions to any 2 parts of Problem (D2) involving exponential generating functions.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose $c_0 = 1$ and

$$c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$$

for $n \geq 0$. Prove that the generating function $C(z)$ of $c_n$ satisfies

$$C(z) = \frac{1 + \sqrt{1 - 4z}}{2z},$$

and use it to derive the closed formula

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$