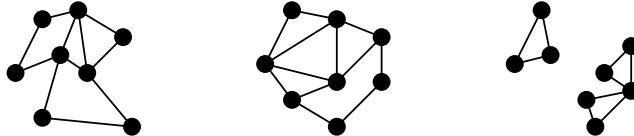


Spring 2022, Math 579: Week 10 Problem Set
Due: Thursday, April 14th, 2022
Introduction to Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Eulerian walks.* Fix a graph $G = (V, E)$. A walk on G is *Eulerian* if every edge in G is used exactly once.

- (a) Which of the following graphs have *closed* Eulerian walk (that is, an Eulerian walk that begins and ends at the same vertex)? Justify your answers.



- (b) Have each member of your group draw a graph with at least 8 vertices and 15 edges. Together, determine which of these graphs has a closed Eulerian walk.
 (c) Conjecture a criterion for when a simple graph G has a closed Eulerian walk.
 (d) Which simple graphs have a (not necessarily closed) Eulerian walk? Try a few first!
 (e) Make each of the above graphs *directed* by manually adding a direction to each edge. Now, which graphs have a closed Eulerian walk? Do this several times for each graph.
 (f) Conjecture a criterion for when a simple directed graph G has a closed Eulerian walk.

(D2) *Graph automorphisms.* An *automorphism* of a graph G is a bijection $f : V(G) \rightarrow V(G)$ such that $(v, w) \in E(G)$ precisely when $(f(v), f(w)) \in E(G)$.

- (a) Draw the cycle graph C_4 , and label the vertices with 1, 2, 3, 4 in a clockwise fashion. Write all 8 automorphisms of C_4 . As an example, one automorphism f is given by

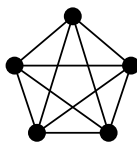
$$f(1) = 2, \quad f(2) = 3, \quad f(3) = 4, \quad \text{and} \quad f(4) = 1.$$

- (b) Complete the following proof that for each $n \geq 3$, the n -vertex cycle graph $G = C_n$ has exactly $2n$ automorphisms. Draw an accompanying “graph fragment” picture too.

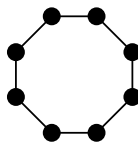
Proof. Label the vertices of G by $1, 2, \dots, n$ in a clockwise fashion. For convenience, if we refer to vertex $n + 1$, we mean vertex 1, and similarly for vertex $n + 2$, etc.

Suppose $f : V(G) \rightarrow V(G)$ is an automorphism of G , and let $v = f(1)$. We must have $f(2) = v + 1$ or $______$ since $(1, 2) \in E(G)$ and these are the only vertices connected to v . If $f(2) = v + 1$, then $f(3) = ______$, $f(4) = ______$, and so on since $______$. Similarly, if $f(2) = ______$, the remaining values of f must again follow in cyclic order. This means f is determined by choosing the value $f(1)$ ($______$ possibilities) and then an adjacent value for $f(2)$ ($______$ possibilities), yielding $2n$ total automorphisms. \square

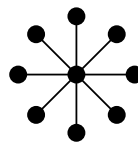
- (c) Prove that if $n \geq 2$ and $G = K_n$ is the complete graph with vertex set $[n]$, then **every** bijection $V(G) \rightarrow V(G)$ is an automorphism.
 (d) Find (with proof!) all automorphisms of the star graph $G = S_n$ with $n \geq 2$ appendages.
 (e) Find (with proof!) all automorphisms of the wheel graph $G = W_n$ for $n \geq 3$ spokes.



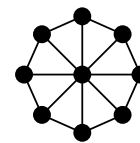
Complete graph K_5



Circle graph C_8



Star graph S_8



Wheel graph W_8

Homework problems. You must submit *all* homework problems in order to receive full credit.

- (H1) Prove that if there is a walk from a vertex v_1 to a vertex v_2 in a simple graph G , then there is a path from v_1 to v_2 .
- (H2) Is there a disconnected simple graph on 7 vertices such that every vertex has degree at least 3?
- (H3) Suppose G is a k -regular graph (that is, a simple graph in which every vertex has degree exactly k). Prove that G has a cycle of length at least $k + 1$.
- (H4) We say two graphs G and G' are *isomorphic* if there exists a bijection $f : V(G) \rightarrow V(G')$ such that $(v, w) \in E(G)$ precisely when $(f(v), f(w)) \in E(G')$.
Find all non-isomorphic simple graphs on four vertices. Be sure to prove (i) no two graphs in your list are isomorphic to one another, and (ii) every four vertex graph is isomorphic to one on your list.
- (H5) Find a simple graph G that has no nontrivial automorphisms (that is, where the only automorphism is the identity map).
- (H6) A *Hamiltonian cycle* is a cycle which visits every vertex exactly once. How many distinct Hamiltonian cycles does the complete graph K_n have?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

- (C1) If G is a simple graph with n vertices and no Hamiltonian cycles, then what is the maximum number of edges G can have? (Your answer should depend on n .)