Spring 2022, Math 579: Week 12 Problem Set
Due: Thursday, April 28th, 2022
Colorings and Bipartite Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Chromatic polynomials.* Fix a graph G with $n = |V(G)|$. The *chromatic function* of G is

$$\chi_G(k) = \#\text{ proper colorings of } G \text{ with } k \text{ colors.}$$

(a) Find the chromatic function $\chi_G(k)$ of each of the following graphs G.

(i) The competely disconnected graph D_n with n vertices and no edges.

(ii) The graph K_5 with one edge removed.

(iii) The complete graph K_n.

(iv) The path graph P_n.

Write each answer as a polynomial in k.

(b) The *complete bipartite graph* is the bipartite graph $K_{n,m} = (X, Y)$ with $|X| = n$ and $|Y| = m$ such that every vertex of X is adjacent to every vertex of Y.

Find the chromatic function $\chi_G(k)$ where $G = K_{2,2}$ and $G = K_{2,3}$.

(c) Let c_i denote the number of ways to properly color G using **exactly** i colors. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \sum_{i=1}^{n} \binom{k}{i} c_i = \binom{k}{1} c_1 + \binom{k}{2} c_2 + \cdots + \binom{k}{n} c_n$$

for all $k \geq 1$. Find c_1, c_2, and c_3 for $G = K_3$, and use this to find $\chi_G(k)$. Verify your answer matches part (a).

(d) Give a combinatorial proof of the identity in part (c).

(e) Using the fact that $\binom{k}{i} = \frac{1}{i!} k(k-1)\cdots(k-i+1)$ is a polynomial in k of degree i, deduce from part (c) that $\chi_G(k)$ is a polynomial in k of degree n.

(f) Fix an edge $e \in E(G)$. Consider (without proof, for the moment) the equality

$$\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k).$$

Using **only** this equality and part (a)(i), find the chromatic polynomial of $G = K_3$.

(g) Prove $\chi_G(k)$ is a polynomial using induction on the number of edges of G. Does this proof guarantee that the coefficients of $\chi_G(k)$ are all integers?

Hint: use part (a)(i) as your base case, and part (f) in your inductive step.

(h) Give a combinatorial proof of the equality in part (f).

(i) An *orientation* of an undirected graph G is a choice of direction for each edge of G. An orientation is *acyclic* if there are no directed cycles.

Find the number of acyclic orientations of the graphs from parts (i), (ii), and (iv) of part (a), and both graphs in part (b). Then, find $(-1)^n \chi_G(-1)$ for each graph. What do you notice?

(j) It turns out $(-1)^n \chi_G(-1)$ equals the number of acyclic orientations of G. Using this fact (without proof, for the moment), find a formula for the number of acyclic orientations of K_n. Then give a direct counting argument for your formula.

(k) Use part (f) and induction on the number of edges of G to prove $(-1)^n \chi_G(-1)$ equals the number of acyclic orientations of G.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the chromatic polynomial of each of the following graphs.

(a) The graph G obtained from K_n by removing one edge.

(b) The graph $K_{2,n}$.

(H2) Fix a graph G with n vertices and m edges.

(a) Prove that the leading coefficient of $\chi_G(k)$ (that is, the coefficient of k^n) is 1.

(b) Prove that the coefficient of k^{n-1} in the chromatic polynomial $\chi_G(k)$ equals $-m$.

Hint: each part of this problem can be proven in (at least) 2 distinct ways. One way uses a direct proof with Problem (D1)(c), and another uses induction with Problem (D1)(f).

(H3) Fix a connected graph G with n vertices. Prove that G is a tree if and only if

$$\chi_G(k) = k(k-1)^{n-1}.$$

Hint: use induction for the forward direction and Problem (H2) for the backwards direction.

(H4) Which of the following can be the degrees of the vertices of a bipartite graph?

(a) $3, 3, 3, 3, 3, 3$ (6 vertices total)

(b) $3, 3, 3, 3, 3, 3, 3, 3$ (8 vertices total)

(c) $3, 3, 3, 3, 3, 5, 6, 6, 6$ (9 vertices total)

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove $(-1)^n\chi_G(-1)$ equals the number of acyclic orientations of G.

Note: see Problems (D1)(i)-(k) for relevant definitions.