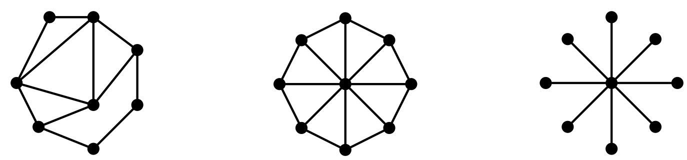
Spring 2022, Math 579: Week 13 Problem Set
Due: Thursday, May 4th, 2022
Planar Graphs

Discussion problems. The problems below should be worked on in class.

(D1) *Counting faces of planar graphs.* For a planar graph G, let V, E, and F denote the number of vertices, edges, and faces of G, respectively.

(a) Compute the quantity $V - E + F$ for each of the following graphs.



(b) Have each group member draw their favorite connected planar graph with at least 8 vertices and 15 edges, and compute $V - E + F$ for their graph.

(c) Notice this came out the same for each graph. This is known as *Euler's theorem* for planar, connected graphs. We will prove this by induction on E.

(i) Base case: prove Euler's theorem when $E = V - 1$. Why is this the base case?
(ii) **Carefully and precisely**, write the inductive hypothesis.
(iii) What can happen when an edge $e \in E(G)$ is removed?
(iv) Finish your proof that Euler's theorem holds for any planar graph G.

(D2) *Duals of planar graphs and a test for planarity.*

(a) Use Euler's Theorem to give a non-pictorial proof that K_5 is not planar.
Hint: how many faces would it have, and how sides would each face need to have?

(b) Use Euler's Theorem to give a non-pictorial proof that $K_{3,3}$ is not planar.
Hint: is it possible for a face to have 3 boundary edges?

(c) Fix a **simple** graph G with V vertices and E edges.

(a) Prove that if G is planar, then $3F \leq 2E$.
Hint: what can be said about vertex degrees in G^*?
(b) Use the previous part and Euler's theorem to prove if G is planar, then $E \leq 3V - 6$.
(c) Is it true that any connected graph satisfying $E \leq 3V - 6$ is planar?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that if any 2 edges are removed from the graph K_6, the result is not planar. Is the same true if we remove 3 edges?

(H2) Suppose for a given **connected** planar graph G, each face of G (including the "outside" one!) has either 3 or 5 boundary edges. Prove that the number of faces of G is even.

Clarification: the *number of boundary edges* of a face F is the number of edges traversed when walking around the boundary of F.

(H3) Suppose G is a connected planar graph in which every face has at least 4 boundary faces. Prove $E \leq 2V - 4$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let T be a triangle partitioned by line segments into a collection $\mathcal{T}$ of smaller triangles (this is called a *triangulation* of T). Assume that the intersection of any 2 triangles in $\mathcal{T}$ is either a full edge of both triangles, a vertex of both triangles, or empty (in particular, no edge of any triangle in $\mathcal{T}$ can contain a vertex of another triangle).

Next, color each vertex red, green, or blue, subject to the following restrictions:

(i) the 3 vertices of T are distinct colors; and

(ii) any vertex on a boundary edge of T is a different color than that of the opposite vertex of T (e.g., any vertex on the edge between the green and blue vertices of T must be colored green or blue).

Prove there is a triangle in $\mathcal{T}$ whose vertices are distinct colors.