Spring 2022, Math 621: Week 1 Problem Set
Due: Thurday, February 3rd, 2022
Numerical and Affine Semigroups

Discussion problems. The problems below should be worked on in class.

(D1) *Minimal generating sets.* Fix a subsemigroup $S \subset (\mathbb{Z}^d_{\geq 0}, +)$. The goal of this problem is to prove that S has a unique generating set that is minimal with respect to containment.

(a) Determine the unique minimal generating set of each of the following semigroups.

(i) $S = \langle 6, 9, 15, 20, 26, 42, 55 \rangle$

(ii) $S = \langle 11, 15, 19, \ldots, 111 \rangle$

(iii) $S = \langle (0,2), (1,2), (2,0), (2,2), (2,3), (3,2), (3,3) \rangle$

(iv) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : a \leq 3b \text{ and } b \leq 4a\}$

(v) $S = \langle G \rangle$, where $G = \{(a,b) : a \equiv 1 \bmod 3 \text{ and } b \equiv 2 \bmod 3\}$.

(b) Let $S^* = S \setminus \{0\}$. Compare the sets S and $S^* + S^*$ for $S = \langle 3, 5 \rangle$.

(c) Let $\mathcal{A}(S) = S^* \setminus (S^* + S^*)$. Prove that $S = \langle \mathcal{A}(S) \rangle$.

(d) Explain briefly why any generating set for S must contain $\mathcal{A}(S)$.

(e) Prove that if $d = 1$, then the elements of $\mathcal{A}(S)$ must be distinct modulo $m = \min \mathcal{A}(S)$. Conclude that $\mathcal{A}(S)$ must be finite in this case.

(f) Must $\mathcal{A}(S)$ be finite if $d \geq 2$?

(D2) *Geometry of semigroups.* The goal of this problem is to explore several geometric theorems.

(a) It turns out any finitely generated subsemigroup $S = \langle \alpha_1, \ldots, \alpha_k \rangle \subset \mathbb{Z}^r$ is isomorphic to some subsemigroup of $\mathbb{Z}^d_{\geq 0}$, where $d = \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}(S)$ (the *affine dimension* of S).

For each of the following, compute d, and locate an isomorphic subsemigroup of $\mathbb{Z}^d_{\geq 0}$.

(i) $S = \langle (9,6), (15,10), (21,14) \rangle$

(ii) $S = \langle (0,2), (2,1), (-1,2) \rangle$

(iii) $S = \langle (2,0,2), (2,3,5), (4,3,7) \rangle$

(b) *Gordan's Lemma*: given a finite list of linear inequalities with rational coefficients, the subset of $\mathbb{Z}^d$ satisfying them is a finitely generated semigroup.

Determine the (finite) minimal generating set of each of the following.

(i) $S = \{(a,b) \in \mathbb{Z}^2 : 0 \leq 3a \leq 2b\}$

(ii) $S = \{v \in \mathbb{Z}^3 : Av \leq 0\}$, where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

and each row of $Av \leq 0$ is interpreted as an inequality on v_1, v_2, v_3.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Fix a commutative semigroup $(S, +)$. A relation $\sim$ on S is a *congruence* if (i) $\sim$ is an equivalence relation, and (ii) $\sim$ is closed under *translation* (i.e., $a \sim b$ implies $a + c \sim b + c$ for all $a, b, c \in S$).

(a) Given a collection $\sim_1, \ldots, \sim_r$ of congruences on S, define the *common refinement* $\sim$ by $a \sim b$ whenever $a \sim_i b$ for all i (we often write $\sim \, = \bigcap_i \sim_i$). Prove that $\sim$ is a congruence on S.

(b) Prove that if $\sim$ is a congruence on S, then the set $S/\sim$ of equivalence classes of $\sim$ is a semigroup under the operation $[a] + [b] = [a + b]$. Be sure to prove well-definedness!

(c) Given a semigroup homomorphism $\varphi : S \to T$, the *kernel* of φ, denoted $\ker \varphi$, is the relation $\sim$ on S setting $a \sim b$ whenever $\varphi(a) = \varphi(b)$. Prove $\ker \varphi$ is a congruence on S.

(d) State and prove a version of the first isomorphism theorem for semigroups.

(H2) Fix a field $\Bbbk$, and let $R = \Bbbk[x_1, \ldots, x_k]$. In what follows, for $a \in \mathbb{Z}^k_{\ge 0}$, we use the shorthand

$$x^a = x_1^{a_1} \cdots x_k^{a_k}.$$

A *(unital) binomial* is a polynomial of the form $x^a - x^b \in R$ for some $a, b \in \mathbb{Z}^k_{\ge 0}$.

(a) Fix an ideal $I \subset R$. Define a relation $\sim_I$ on $\mathbb{Z}^k_{\ge 0}$ by

$$a \sim_I b \qquad \text{whenever} \qquad x^a - x^b \in I$$

for $a, b \in \mathbb{Z}^k_{\ge 0}$. Prove that $\sim_I$ is a congruence on $\mathbb{Z}^k_{\ge 0}$.

(b) Consider the semigroup homomorphism $\varphi : \mathbb{Z}^3_{\ge 0} \to \mathbb{Z}_{\ge 0}$ given by

$$(a, b, c) \longmapsto 4a + 5b + 6c.$$

Locate an ideal $I \subset \Bbbk[x, y, z]$ such that $\sim_I \, = \ker \varphi$. Find a finite generating set for I.

(c) Consider the semigroup homomorphism $\varphi : \mathbb{Z}^5_{\ge 0} \to \mathbb{Z}_{\ge 0}$ given by

$$(a_1, a_2, a_3, a_4, a_5) \longmapsto 15a_1 + 97a_2 + 152a_3 + 153a_4 + 201a_5.$$

Locate an ideal $I \subset \Bbbk[x_1, x_2, x_3, x_4, x_5]$ such that $\sim_I \, = \ker \varphi$.
Hint: do **not** attempt to write down generators for I! Instead, specify I as the kernel of a carefully chosen ring homomorphism.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) For any $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}^d_{\ge 0}$ with $d \ge 2$, the set $\langle \alpha_1, \ldots, \alpha_k \rangle \setminus \langle \alpha_1, \ldots, \alpha_{k-1} \rangle$ is infinite.

(b) Given any congruence $\sim$ on $\mathbb{Z}^k_{\ge 0}$, the semigroup $S = \mathbb{Z}^k_{\ge 0}/\sim$ is *cancellative*, that is, $a + c = b + c$ implies $a = b$ for all $a, b, c \in S$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a commutative, cancellative semigroup S such that (i) S is finitely generated, (ii) $0 \in S$ is the only element of S with an inverse, and (iii) S is not isomorphic to any affine semigroup.