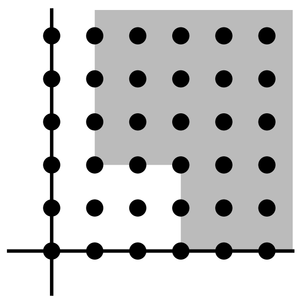
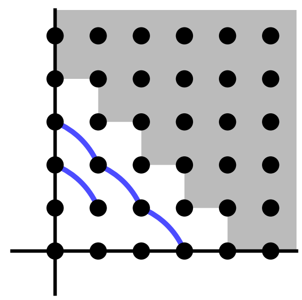
Spring 2022, Math 621: Week 2 Problem Set
Due: Thurday, February 10th, 2022
Graded Rings and Hilbert Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Combinatorics of monomial and binomial quotients.* The goal of this problem is to make sense of pictures of the following form (called *staircase diagrams*).



$J_1 = \langle x^3, xy^2\rangle$



$J_2 = \langle x^3 - x^2y, xy - y^2, x^4\rangle$

(a) Using J_1 above as a guide, draw the staircase diagram for $I = \langle x^3y, x^3y^3, xy^3\rangle$.

(b) For an ideal I with monomial and binomial generators, we add *edges* to the staircase diagram so each connected component corresponds to an equivalence class modulo $\sim_I$. Locate the largest monomial ideal contained in J_2.

(c) Using J_2 above as a guide, draw the staircase diagram of each of the following ideals I, and find $\dim_{\Bbbk} R/I$.

(i) $I = \langle x - y, x^3\rangle$

(ii) $I = \langle y(x^2 - 1), xy^2 - y^2, y^3\rangle$

(iii) $I = \langle x^3 - 1, y(x^2 - 1), y^2(x - 1), y^3\rangle$

(d) Draw the staircase diagram for $I = \langle x^3, y^2, z^2, xyz\rangle \subset \Bbbk[x, y, z]$ (yes, a 3D picture).

(e) Does there exist an ideal $J \supsetneq I$, generated by monomials and binomials, for which I is the largest monomial ideal contained in J?

(D2) *Computing Hilbert functions.* Let $R = \Bbbk[x, y]$, graded by total degree. For each of the following ideals $I \subseteq R$, determine the Hilbert function of I and that of R/I. Your answer to each should be (possibly piecewise) formulas for $\mathcal{H}(I; t)$ and $\mathcal{H}(R/I; t)$ in terms of t.

(a) $I = J_1$ from (D1)

(b) $I = \langle x^3, y^3\rangle$

(c) $I = J_2$ from (D1)

(d) $I = \langle x^2 - y^2\rangle$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find a formula for the Hilbert function of each of the following graded quotient rings R/I.

(a) $R = \mathbb{k}[x, y]$ and $I = \langle x^5 - y^3 \rangle$, under the grading $\deg(x) = 3$, $\deg(y) = 5$

(b) $R = \mathbb{k}[x, y, z]$ and $I = \langle x^3, y^3, z^3, xy^2, yz^2, x^2z \rangle$, under the standard grading

(H2) Find $\dim_{\mathbb{k}}(R/I)$, where $R = \mathbb{k}[x, y]$ and $I = \langle x^3 - y^2, x^5 - y^4 \rangle$.

(H3) Fix an ideal $I \subset R = \mathbb{k}[x_1, \ldots x_k]$. Develop a criterion for when I is homogeneous under **all** $\mathbb{Z}$-gradings of R (that is to say, for any choices of $\deg(x_1), \ldots, \deg(x_k) \in \mathbb{Z}_{\geq 0}$, the ideal I is homogeneous).

(H4) Fix a ring R graded by a semigroup T, and a homogeneous ideal $I \subseteq R$.

(a) Prove that I inherets a grading from R, that is, $I = \bigoplus_t I_t$ for some $\mathbb{k}$-vector spaces $I_t \subseteq I$ with $I_t I_r \subset I_{t+r}$.

(b) Prove that if for each $t \in \mathbb{Z}_{\geq 1}$, V_t is a $\mathbb{k}$-vector space and $U_t \subseteq V_t$ is a subspace, then

$$\left(\bigoplus_{t\geq 1} V_t\right) \Big/ \left(\bigoplus_{t \geq 1} U_t\right) \cong \bigoplus_{t\geq 1} (V_t/U_t).$$

Conclude that the ring R/I also inherets a grading from R, and that if R is positively graded, then

$$\mathcal{H}(R/I; t) = \mathcal{H}(R; t) - \mathcal{H}(I; t).$$

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) For any $a \in \mathbb{Z}_{\geq 0}$, there exists an ideal $I \subseteq R = \mathbb{k}[x, y, z]$, homogeneous under the standard grading, such that $\mathcal{H}(R/I; t) = at + 1$.

(b) If $I \subseteq J$ are homogeneous ideals in $R = \mathbb{k}[x_1, \ldots, x_k]$ (under the standard grading) and $\mathcal{H}(R/I; t) = \mathcal{H}(R/J; t)$, then $I = J$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: if $I \subset J$ are homogeneous ideals in $R = \mathbb{k}[x_1, \ldots, x_k]$ (under the standard grading) and $\sim_I = \sim_J$, then $I = J$.