Spring 2022, Math 621: Week 4 Problem Set
Due: Thurday, February 24th, 2022
Graded Modules

Discussion problems. The problems below should be worked on in class.

(D1) Draw a staircase diagram for each of the following modules M over $R = \Bbbk[x, y]$.

- (a) $M = R$
- (b) $M = I$, where $I = \langle x^3, y^2 \rangle \subseteq R$
- (c) $M = R/I$, where $I = \langle x^3, y^2 \rangle \subseteq R$
- (d) $M = R/I$, where $I = \langle x^3 - y^2 \rangle \subseteq R$
  Remember: monomials in I are represented differently from binomials in I!
- (e) $M = R/I$, where $I = \langle x^3 - y^2, y^4 \rangle \subseteq R$
- (f) $M = R \oplus R$
- (g) $M = R \oplus (R/I)$, where $I = \langle xy^2 \rangle \subseteq R$.
- (h) $M = (R \oplus R)/N$, where $N = \langle xe_1 - ye_2 \rangle \subseteq R \oplus R$.
  (here, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis of $R \oplus R$)
- (i) $M = (R \oplus R)/N$, where $N = \langle xe_1 - ye_2, xy^2e_2 \rangle \subseteq R \oplus R$

(D2) *Computing Hilbert functions.* Let $R = \Bbbk[x, y]$, graded by total degree. Determine the Hilbert function of each of the following graded modules M. Your answer to each should be a (possibly piecewise) formula for $\mathcal{H}(M; t)$ in terms of t.

- (a) $M = I$, where $I = \langle x^2y \rangle \subseteq R$
  What is the relationship between $\mathcal{H}(M; t)$ and $\mathcal{H}(R; t)$ in this example?
- (b) $M = I$, where $I = \langle x^2y, xy^2 \rangle \subseteq R$
- (c) $M = R/I$, where $I = \langle x^2y, xy^2 \rangle \subseteq R$
- (d) $M = R \oplus (R/I)$, where $I = \langle x^2 + xy + y^2 \rangle \subseteq R$
- (e) $M = (R \oplus R)/N$, where $N = \langle x^2e_1 - xye_1, xye_2 - y^2e_2 \rangle \subseteq R \oplus R$
- (f) $M = (R \oplus R)/N$, where $N = \langle x^2e_1 - xye_2, xye_1 - y^2e_2 \rangle \subseteq R \oplus R$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each function h below, find a graded ring R and a finitely generated graded module M over R with $\mathcal{H}(M;t) = h(t)$ for all t in the domain of h.

(a) $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by $h(t) = 5t + 7$

(b) $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by $h(t) = t^2$

(c) $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by

$$h(t) = \begin{cases} t - 7 & \text{if } t \geq 10; \\ 3 & \text{if } t = 6; \\ 0 & \text{otherwise} \end{cases}$$

(d) $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by

$$h(t) = \begin{cases} t + 1 & \text{if } t \equiv 0 \bmod 3; \\ 17 & \text{if } t \equiv 1 \bmod 3; \\ t + 13 & \text{if } t \equiv 2 \bmod 3 \end{cases}$$

(e) $h : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by

$$h(t_1, t_2) = \begin{cases} 2 & \text{if } t_1 = 0 \text{ or } t_2 = 0; \\ t_1 + t_2 & \text{otherwise} \end{cases}$$

(H2) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Any function $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ occurs as the Hilbert function of some graded module over a polynomial ring.

(b) Any function $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ occurs as the Hilbert function of some finitely generated graded module over a polynomial ring.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: there exists a finitely generated, graded module M over $R = \Bbbk[x, y, z, w]$ (under the standard grading) such that $\mathcal{H}(M;t) = t^3$.