Spring 2022, Math 621: Week 5 Problem Set
Due: Thursday, March 3rd, 2022
Rational Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Power series of quasipolynomial functions.* Recall that in lecture, we saw

$$1 + 2z + 3z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2},$$

and that the "formal derivative" of $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is

$$A'(z) = \tfrac{d}{dz}A(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n.$$

(a) Manipulate the first expression to write $\sum_{n=0}^{\infty} nz^n$ as a rational expression in z.
(b) Use formal differentiation to write $\sum_{n=0}^{\infty} n^2 z^n$ as a rational expression in z.
(c) Use formal differentiation to write $\sum_{n=0}^{\infty} n^3 z^n$ as a rational expression in z.

(D2) *Multivariate power series.* In this problem, we will explore a geometric interpretation of rational power series in the ring $\mathbb{Q}[[z_1, z_2]]$.

(a) Using power series multiplication, find all nonzero terms in

$$A(z) = \frac{1}{(1-z_1^3 z_2)(1-z_2^2)}$$

with total degree at most 10. Plot their exponents as points in $\mathbb{R}^2$.

(b) Do the same for the power series

$$B(z) = \frac{1}{(1-z_1^2)(1-z_1 z_2)(1-z_2^2)}.$$

Label each point with its coefficient in $B(z)$. What does this appear to coincide with?

(c) Find a rational expression for the formal power series

$$C(z) = \sum_{(a,b)\in S} z_1^a z_2^b$$

for each of the following sets $S \subset \mathbb{Z}^2_{\geq 0}$.

(i) $S = \langle (0,2), (1,1), (0,2) \rangle$
(ii) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b\}$
(iii) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b \text{ and } a \geq 2\}$
(iv) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : x^a y^b \in I\}$, where $I = \langle x^3, x^2y, y^2 \rangle \subset \Bbbk[x,y]$
(v) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : x^a y^b \notin I\}$, where $I = \langle x^3, x^2y, y^2 \rangle \subset \Bbbk[x,y]$
(vi) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : \mathcal{H}(R/I; a,b) \neq 0\}$, where $I = \langle x_1^2 - x_2^2, x_3^3 \rangle \subset R = \Bbbk[x_1,x_2,x_3]$ with $\deg(x_1) = \deg(x_2) = (1,0)$ and $\deg(x_3) = (0,1)$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each of the following, find a rational expression for the formal power series

$$C(z) = \sum_{(a,b)\in S} z_1^a z_2^b.$$

(a) $S = \{(a,b) \in \mathbb{Z}_{\geq 0}^2 : a \leq 2b,\ b \leq 2a, \text{ and } a + b \geq 3\} \subseteq \mathbb{Z}_{\geq 0}^2$

(b) $S = \{(a,b) \in \mathbb{Z}_{\geq 0}^2 : a \leq 2b + 2,\ b \leq 2a + 2, \text{ and } a + b \geq 2\} \subseteq \mathbb{Z}_{\geq 0}^2$

(c) $S = \{(a,b) \in \mathbb{Z}_{\geq 0}^2 : x^a y^b \in I\}$, where $I = \langle x^6, x^4 y, x^2 y^3, y^5\rangle \subseteq \Bbbk[x,y]$

(H2) Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{Q}$ is a function, $h(z)$ is a power series, $d \in \mathbb{Z}_{\geq 0}$, and

$$\sum_{n=0}^{\infty} f(n) z^n = \frac{h(z)}{(1-z)^{d+1}}.$$

(a) Prove that for any $d \geq 0$,

$$\sum_{n=0}^{\infty} n^d z^n = \frac{h_d(z)}{(1-z)^{d+1}}$$

for some polynomial $h_d(z)$ of degree d with $h_d(1) \neq 0$ (by "polynomial" here, we mean $h_d(z)$ is a power series with finitely many nonzero terms).

Hint: you may use (free of charge) that differentiation of power series respects Calculus 1 derivative rules.

(b) Prove $f(n)$ is a polynomial of degree at most d if and only if $h(z)$ is a polynomial in z with $\deg h(z) \leq d$.

(c) Prove moreover that $f(n)$ has degree exactly d if and only if $h(1) \neq 0$.

(d) We say $f(n)$ is *eventually polynomal* if there exists $N \in \mathbb{Z}_{\geq 0}$ and a polynomial $g(n)$ such that $f(n) = g(n)$ for all $n \geq N$.

Prove that $f(n)$ is eventually polynomial with degree exactly d if and only if $h(z)$ is a polynomial with $h(1) \neq 0$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Characterize which functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ satisfy

$$\sum_{n \geq 0} f(n) z^n = \frac{h(z)}{g(z)}$$

for some polynomials $h(z)$ and $g(z)$ with coefficients in $\mathbb{C}$.