Spring 2022, Math 621: Week 7 Problem Set
Due: Thursday, March 17th, 2022
Ehrhart Polynomials and Ehrhart Series

Discussion problems. The problems below should be worked on in class.

(D1) *Computing Ehrhart functions.* Find the Ehrhart function of each polytope **without** using Ehrhart's theorem.

(a) The lattice polygon $P = \operatorname{conv}\{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\} \subseteq \mathbb{R}^2$.

(b) The lattice simplex $P = \operatorname{conv}\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)\} \subseteq \mathbb{R}^3$.

(D2) *A geometric proof of Ehrhart's theorem.*

(a) We begin by considering the following theorem.

Theorem. *Any rational cone $C \subset \mathbb{R}^d$ with linearly independent extremal ray vectors $r_1, \ldots, r_k \in \mathbb{Z}^d_{\geq 0}$ has (multivariate) power series*

$$\sum_{p \in C \cap \mathbb{Z}^d} z^p = \frac{h(z)}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}$$

for some polynomial $h(z)$.

As is always **strongly** suggested, we begin with a couple of examples.

(i) Let $C = \operatorname{span}_{\geq 0}\{(-\frac{1}{2}, 1), (2, 1)\} \subseteq \mathbb{R}^2_{\geq 0}$ denote the cone over $P = [-\frac{1}{2}, 2] \subseteq \mathbb{R}$. Find the multivariate power series for C. How is this related to the Ehrhart series for P? What role is the "fundamental parallelopiped" of C playing?

(ii) Let $C \subseteq \mathbb{R}^3_{\geq 0}$ denote the cone over $P = \operatorname{conv}\{(0,0), (2,1), (3,2)\}$. Draw P, $2P$, and $3P$, and list a few integer points in each corresponding slice of C. Then find the multivariate power series for C and the Ehrhart series for P.

Now, outline a proof for the above theorem.

(b) Let $C \subset \mathbb{R}^3$ denote the cone over the unit square $P = [0,1]^2$. Divide P into two triangles by drawing in one of the two diagonals. This is called a *triangulation* of P. **Using the above theorem**, demonstate there is a polynomial $h(z)$ such that

$$\operatorname{Ehr}(P; z) = \frac{h(z)}{(1-z)^3}$$

by writing $\operatorname{Ehr}(P; z)$ in terms of the Ehrhart series of 3 simplices.

(c) Use triangulation to find the Ehrhart series of

$$P = \operatorname{conv}\{e_1, -e_1, e_2, -e_2, e_3, -e_3\} \subseteq \mathbb{R}^3.$$

Hint: utilize symmetry by dividing P into 8 tetrahedra.

(d) Briefly justify the following theorem (a classical result from polyhedral geometry) in the special case where P is an arbitrary rational polygon.

Theorem. *Any rational polytope P can be written as the union of rational simplices $T_1, \ldots, T_r$ such that (i) the vertices of each T_i coincide with vertices of P, (ii) each intersection $T_i \cap T_j$ is a face of both T_i and T_j, and (iii)* $\operatorname{vol} T_1 + \cdots + \operatorname{vol} T_r = \operatorname{vol} P$.

(e) Putting everything together, outline a proof for Ehrhart's theorem: for any d-dimensional rational polytope P, we have

$$\operatorname{Ehr}(P; z) = \frac{h(z)}{(1 - z^p)^{d+1}}$$

for some $p \geq 1$ and some polynomial $h(z)$. How is the value of p determined from P?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove $f(n)$ is eventually quasipolynomial with period p and degree at most d if and only if

$$\sum_{n\geq 0} f(n)z^n = \frac{h(z)}{(1-z^p)^{d+1}}$$

for some polynomial $h(z)$.

(H2) Find the Ehrhart function and Ehrhart series of the rational polygon

$$P = \operatorname{conv}\{(1,0),\ (1,\tfrac{1}{2}),\ (\tfrac{1}{2},1)\} \subseteq \mathbb{R}^2.$$

(H3) Prove the following. You may use any theorems we have seen involving Ehrhart polynomials, including ones we have not (yet) proven.

Theorem (Pick's Theorem)**.** *For any lattice polygon P with I interior lattice points, B boundary lattice points, and area A, we have*

$$A = I + \tfrac{1}{2}B - 1.$$

(H4) Find, for each $h \in \mathbb{Z}_{\geq 1}$, the Ehrhart polynomial and Ehrhart series of

$$P = \operatorname{conv}\{(0,0,0),\ (0,0,1),\ (0,1,0),\ (h,1,1)\}$$

(this is known as *Reeve's tetrahedron*). You may use any theorems we have seen involving Ehrhart polynomials, including ones we have not (yet) proven.

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Pick's theorem also holds for rational polygons.

(b) For each $h \geq 1$, there exists an integral polytope $P \subseteq \mathbb{R}^2$ such that

$$\operatorname{Ehr}(P;z) = \frac{1+hz+z^2}{(1-z)^3}.$$