**Spring 2022, Math 621: Week 8 Problem Set**
**Due: Thursday, March 24th, 2022**
**Hilbert Series and Hilbert's Theorem**

**Discussion problems.** The problems below should be worked on in class.

(D1) *Ehrhart from Hilbert.* The goal of this problem is to link Ehrhart's theorem and Hilbert's.

(a) As is always **strongly** suggested, we begin with an example. Let $C \subset \mathbb{R}^3_{\geq 0}$ denote the cone over the simplex $P = \text{conv}\{(0,0),(2,1),(3,2),(0,3)\}$, and let $S = C \cap \mathbb{Z}^3$ denote the associated semigroup.

(b) Draw $P$, $2P$, and $3P$. List a few integer points in each corresponding slice of $C$.

(c) Find all 9 minimal generators of $S$.

(d) Applying Hilbert's theorem to $\Bbbk[S] \subset \Bbbk[x,y,z]$ under the fine grading, what form do we obtain for $\text{Hilb}(\Bbbk[S]; z_1, z_2, z_3)$ **without** computing a precise numerator?

(e) What grading should we choose in the previous part to instead obtain $\text{Ehr}(P; z)$?

(f) Explain why $\Bbbk[x,y,z]$ is a module over $\Bbbk[S]$. Is it finitely generated?

(g) Explain why $M = \Bbbk[S]$ is a module over $R = \Bbbk[z, x^2yz, x^3y^2z, y^3z]$, and find all 7 minimal homogeneous generators of $M$ (as an $R$-module).

(h) Apply Hilbert's theorem to the module $M$. What does this tell us about the rational form of the Ehrhart series of $P$?

(D2) *Hilbert series of monomial ideals.*

(a) Find $\text{Hilb}(M; z_1, z_2)$ for each of the finely-graded modules $M$ over $R = \Bbbk[x,y]$ below.
   (i) $M = I$, where $I = \langle xy^2 \rangle$
   (ii) $M = I$, where $I = \langle x^3, xy^2, y^4 \rangle$
   (iii) $M = R/I$, where $I = \langle x^3, xy^2, y^4 \rangle$
   (iv) $M = R/I$, where $I = \langle x^4, x^3y, x^2y^2, y^4 \rangle$

(b) Find $\text{Hilb}(M; z)$ for each of the above modules $M$ (under the standard grading).

(c) Conjecture a closed form for $\text{Hilb}(R/I; z_1, z_2)$, where

$$I = \langle x^{a_1}y^{b_1}, \ldots, x^{a_k}y^{b_k} \rangle \subset R = \Bbbk[x,y]$$

is a monomial ideal with $a_1 > \cdots > a_k$ and $b_1 < \cdots < b_k$.

(d) Locate a monomial ideal $I \subset \Bbbk[x,y]$ for which, under the standard grading,

$$\text{Hilb}(I; z) = \frac{3z^5 - z^6 + z^7 - 2z^8}{(1-z)^2}.$$

(e) Find $\text{Hilb}(R/I; z_1, z_2, z_3)$, where $I = \langle x^2, y^2, z^2, xyz \rangle \subset R = \Bbbk[x,y,z]$.
Hint: draw the staircase diagram (yes, in 3D).

(f) Locate a monomial ideal $I \subset \Bbbk[x_1, x_2, x_3]$ for which, under the standard grading,

$$\text{Hilb}(I; z) = \frac{6z^2 - 8z^3 + 3z^4}{(1-z)^3}.$$

**Homework problems.** You must submit *all* homework problems in order to receive full credit.

(H1) Locate a **single** graded ring $R$ and finitely generated $R$-modules $M_1$, $M_2$, and $M_3$ with the Hilbert series specified below.

(a) $\operatorname{Hilb}(M_1; z) = \dfrac{1}{(1-z^3)(1-z^5)}$

(b) $\operatorname{Hilb}(M_2; z) = \dfrac{z^5 + 3z^7}{(1-z^2)(1-z^3)^2}$

(c) $\operatorname{Hilb}(M_3; z) = \dfrac{2z^5 - z^7}{(1-z^2)(1-z^3)}$

(H2) Fix a rational polyhedron $R$, and for simplicity, assume $R \subset \mathbb{R}^d_{\geq 0}$. It is known that $R = P + C$ for some rational polytope $P \subset \mathbb{R}^d_{\geq 0}$ and some rational cone $C \subset \mathbb{R}^d_{\geq 0}$. Use this fact to prove the multivariate generating function of $R$ satisfies

$$\sum_{p \in R \cap \mathbb{Z}^d} z^p = \frac{h(z)}{(1-z^{r_1}) \cdots (1-z^{r_k})}$$

for some polynomial $h(z)$ and some $r_1, \ldots, r_k \in \mathbb{Z}^d_{\geq 0}$.

(H3) The *Hilbert series* of a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}$ is

$$\operatorname{Hilb}(S; z) = \sum_{a \in S} z^a.$$

(a) Prove **using Hilbert's Theorem** that

$$\operatorname{Hilb}(S; z) = \frac{g(z)}{(1-z^{n_1}) \cdots (1-z^{n_k})}$$

for some polynomial $g(z)$.

(b) Fix $m \in S$. Prove that

$$\operatorname{Hilb}(S; z) = \frac{a(z)}{1-z^m}$$

for some polynomial $a(z)$ with positive integer coefficients.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If $I \subset R = \Bbbk[x_1, \ldots, x_k]$ is a monomial ideal, and we write

$$\operatorname{Hilb}(R/I; z_1, \ldots, z_k) = \frac{h(z_1, \ldots, z_k)}{(1-z_1) \cdots (1-z_k)} \qquad \text{and} \qquad \operatorname{Hilb}(R/I; z) = \frac{g(z)}{(1-z)^k}$$

with respect to the fine grading and the standard grading, respectively, then $h$ and $g$ have the same number of nonzero terms.

(b) There exists a monomial ideal $I \subset \Bbbk[x_1, x_2, x_3]$ such that, under the standard grading,

$$\operatorname{Hilb}(I; z) = \frac{3z^6 - 4z^9 + z^{11}}{(1-z)^3}.$$