Spring 2022, Math 621: Week 9 Problem Set
Due: Thursday, April 7th, 2022
Hilbert's Theorem and Quasipolynomials

Discussion problems. The problems below should be worked on in class.

(D1) *Staircase diagrams of modules.* Let $R = \Bbbk[x, y]$, under the **standard grading**.

(a) Draw the staircase diagram of the ideal $I = \langle x^2y - xy^2, x^3, y^3\rangle$, and find $\mathrm{Hilb}(R/I; z)$.
(b) Draw the staircase diagram of $J = \langle x^2 - y^2, x^5 - y^3\rangle$, and use it to find $\dim_\Bbbk R/J$.
(c) Let $I = \langle x^2, y^2\rangle$ and $J = \langle x - y, x^2\rangle$, and let $M = (R/I) \oplus (R/J)$. Find $\dim_\Bbbk M$. Is there a "staircase diagram for M" that helps with this?
(d) Let $N = \langle x^2e_1, y^2e_1, (x - y)e_2, x^2e_2\rangle \subset R^2$, and let $M = R^2/N$. Find $\dim_\Bbbk M$. (here, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis of R^2)
(e) Let $N = \langle (x - y)e_1, (x - y)e_2, xe_1 - ye_2\rangle \subset R^2$, and let $M = R^2/N$. Find $\mathrm{Hilb}(M; z)$. Hint: can you find a drawing that faithfully encodes this information?

(D2) *Homogeneous system of parameters.* Throughout this problem, suppose R is a graded ring and M is a graded R-module. The goal of this problem is to obtain a (more) optimal denominator for the Hilbert series of an R-module M.

(a) An element $r \in R$ is a *nonzerodivisor* on M if $rm = 0$ implies $m = 0$.
Determine which variables in $R = \Bbbk[x, y, z, w]$ are nonzerodivisors on $M = R/\langle x^3, y^5w^2\rangle$.
Do the same for $M = R/\langle x - y, z^3 - zw^5\rangle$.
(b) Let $d = \dim(M)$. We call $r_1, \ldots, r_d \in R$ a *homogeneous system of parameters (HSOP)* for M if for each i, we have r_{i+1} is a nonzerodivisor on $M/(r_1M + \cdots + r_iM)$.
Note: a system of parameters consists of **ring elements**, not module elements.
Verify $r_1 = x, r_2 = y$ is a HSOP for $M = R/\langle x-y\rangle$ over $R = \Bbbk[x, y]$ (standard graded).
(c) Verify the ring $R = \Bbbk[x, y]/\langle xy\rangle$, under the fine grading, does not have a HSOP.
(d) Locate a homogeneous system of parameters for each of the following modules M over $R = \Bbbk[x, y]$ (under the standard grading).
(a) $R/\langle x^2\rangle$ (b) $R/\langle xy\rangle$ (c) $R/\langle x^2y\rangle \oplus R/\langle xy^2\rangle$
(e) The following is a refinement of Hilbert's theorem: if R and M satisfy the hypotheses of Hilbert's theorem, and $r_1, \ldots, r_d \in R$ is a HSOP for M, then

$$\mathrm{Hilb}(M; z) = \frac{h(z)}{(1 - z^{\deg(r_1)}) \cdots (1 - z^{\deg(r_d)})}$$

for some polynomial $h(z)$. Verify this for each module from the previous part.

(D3) *Computing Hilbert series.* Find a rational expression for the Hilbert series $\mathrm{Hilb}(M; z)$ of each of the following graded modules M over $R = \Bbbk[x, y]$ (under the standard grading). Additionally, find $\dim M$ and a homogeneous system of parameters for M, and simplify your rational expression so it matches the one with "smaller" denominator in (D2).

(a) $M = R/I$, where $I = \langle x^2 - y^2\rangle \subset R$
(b) $M = R/I$, where $I = \langle x^3 - xy^2\rangle \subset R$
(c) $M = R \oplus (R/I)$, where $I = \langle x^2 + xy + y^2\rangle \subset R$
(d) $M = (R/I) \oplus (R/J)$, where $I = \langle x^3 - x^2y, xy - y^2\rangle$ and $J = I + \langle x^4\rangle$
(e) $M = R^2/N$, where $N = \langle x^2e_1 - xye_1, xye_2 - y^2e_2\rangle \subset R^2$
(f) $M = R^2/N$, where $N = \langle x^2e_1 - xye_2, xye_1 - y^2e_2\rangle \subset R^2$
(g) $M = R^2/N$, where $N = \langle m\rangle$ for some $m \in R^2$ with $\deg(m) = 17$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Given an R-module M and an element $m \in M$, we define the *annihilator* of m as

$$\operatorname{ann}(m) = \{r \in R : rm = 0\}.$$

Let $R = \Bbbk[x, y]$ and $M = R/I$, where $I = \langle x^2y - xy^2, x^3, y^3\rangle$. Find the set $N \subset M$ of all elements $m \in M$ for which $\operatorname{ann}(m) = \langle x, y\rangle$. What kind of object is N? A $\Bbbk$-vector space? A submodule of M?

(H2) Do any 2 parts of Problem (D3) (c)-(g).

(H3) Do one (or more!) of the following problems.

(a) Let $R = \Bbbk[x, y]$ under the standard grading, and fix a submodule $N = \langle m_1, m_2\rangle \subset R^2$ such that $\deg(m_1) = 17$ and $\deg(m_2) = 11$. Hilbert's theorem ensures

$$\operatorname{Hilb}(R^2/N; z) = \frac{h(z)}{(1 - z)^2}$$

for some polynomial $h(z)$. Find all possible $h(z)$.

(b) Fix a monomial ideal $I \subset \Bbbk[x_1, \ldots, x_k]$, and write

$$\operatorname{Hilb}(I; z) = \frac{h(z)}{(1 - z)^k}$$

as in Hilbert's Theorem (under the standard grading). Find a formula for $h(1)$.

(c) Fix a monomial ideal $I = \langle x^{m_1}, \ldots, x^{m_r}\rangle \subset \Bbbk[x_1, \ldots, x_k]$. Characterize $\dim(R/I)$ in terms of the exponent vectors $m_1, \ldots, m_r$.

(H4) For at least one of the following, determine whether the statements is true or false. Prove your assertion(s).

(a) If $R = \Bbbk[x, y, z]$ is $\mathbb{Z}$-graded and M is a graded R-module such that for $t \gg 0$,

$$\mathcal{H}(M; t) = f(t)$$

for some quasipolynomial function $f(t)$, then $f(t)$ has constant leading coefficient.

(b) If $R = \Bbbk[x, y, z]$ is $\mathbb{Z}$-graded and M is a graded R-module such that for $t \gg 0$,

$$\mathcal{H}(M; t) = f(t)$$

for some quasipolynomial function $f(t)$ with constant leading coefficent c, then it is possible for c to be any positive rational number.

(c) If $R = \Bbbk[x, y, z]$ is $\mathbb{Z}$-graded and M is a graded R-module such that for $t \gg 0$,

$$\mathcal{H}(M; t) = f(t)$$

for some quasipolynomial function $f(t)$ with period p, then there exists a homogeneous system of parameters $r_1, \ldots, r_d \in R$ for M such that $\operatorname{lcm}(\deg(r_1), \ldots, \deg(r_d)) = p$.