Spring 2022, Math 621: Week 10 Problem Set
Due: Thursday, April 14th, 2022
Gröbner Bases

Discussion problems. The problems below should be worked on in class.

(D1) *Gröbner bases of modules.* Let $R = \Bbbk[x_1, \ldots, x_k]$, and let $e_1, \ldots, e_n$ are the standard basis vectors of the free module R^n. The goal of this problem is to extend the concept of a Gröbner basis from ideals in R to submodules of R^n.

(a) Recall that a *monomial* in R^n is an element of the form $x^a e_i$ (that is, a monomial in R times a standard basis vector of R^n). Decide what it means for $x^a e_i$ to divide $x^b e_j$.

(b) Given a term order $\preceq$ on R, define the *position-over-term* (POT) order $\preceq_{\text{pot}}$ on R^n so $x^a e_i \preceq_{\text{pot}} x^b e_j$ whenever (i) $i > j$ or (ii) $i = j$ and $x^a \preceq x^b$. Find the initial terms of
$$m = x^4ye_1 + x^3y^3e_1 + y^2e_1 + x^4ye_2 + xy^3e_2 \qquad \text{and} \qquad m' = xye_1 - x^3e_2$$
under pot-glex order, and use the division algorithm to divide m by m'.

(c) Given a term order $\prec$ on R, define the *term-over-position* (TOP) order $\preceq_{\text{top}}$ on R^n so $x^a e_i \preceq_{\text{top}} x^b e_j$ whenever (i) $x^a \preceq x^b$, or (ii) $x^a = x^b$ and $i > j$. Find the initial terms of m and m' from the previous part under top-glex order, and use the division algorithm to divide m by m'.

(d) Using the above as intuition, decide on a reasonable definition of a *term order* $\prec$ on R^n.

(e) Give a reasonable definition of a *Gröbner basis* $m_1, \ldots, m_r$ of a submodule $M \subset R^n$ with respect to a given term order $\preceq$ on R^n.

(f) Identify a reasonable element in R^2 to serve as the sygyzgy
$$S(x^2ye_1 + x^5e_2, y^3e_1 + x^2e_2 - ye_2)$$
under the pot-glex order. Is there a reasonable choice for $S(x^2e_1 + ye_2, y^2e_2 + xe_2)$? Use your intuition to carefully define the *syzygy* element $S(m, m')$, keep in mind that we want the following theorem to hold.

Theorem. *A list of elements $m_1, \ldots, m_r$ is a Gröbner basis for $M \subset R^n$ if and only if division of each syzygy $S(m_i, m_j)$ by $m_1, \ldots, m_r$ yields remainder 0.*

(g) Compute a Gröbner basis for the submodule
$$M = \left\langle x^2e_1 - y^2e_1,\, xye_1 - ye_2 \right\rangle \subseteq R^2$$
under the pot-glex term order.

(D2) *Reduced Gröbner bases.* Fix an ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_k]$ and a term order $\preceq$.

(a) Argue that if $g_1, \ldots, g_r$ is a Gröbner basis for an ideal I, and g_r' is the remainder after dividing g_r by $g_1, \ldots, g_{r-1}$, then $g_1, \ldots, g_{r-1}, g_r'$ is also a Gröbner basis for I.

(b) A Gröbner basis G is called *reduced* if (i) for all $g_i, g_j \in G$, no term of g_i is divisible by the leading term of g_j, and (ii) the coefficient of each leading term is 1.
Use the previous part to find a reduced Gröbner basis for the ideal
$$J = \langle x^3 - y^2, x^4y^4 - z^3, xy^6 - z^3, y^8 - x^2z^3\rangle \subseteq \Bbbk[x, y, z]$$
under glex order. You may assume the given generating set is a Gröbner basis (it is).

(c) The *initial ideal* of an ideal I under $\preceq$ is the monomial ideal
$$\text{In}_\preceq(I) = \langle \text{In}_\preceq(f) : f \in I\rangle$$
generated by the initial terms of **every** element of I. Argue that $G = \{g_1, \ldots, g_r\}$ is a Gröbner basis for I under $\preceq$ if and only if we have $\text{In}_\preceq(I) = \langle \text{In}_\preceq(g_1), \ldots, \text{In}_\preceq(g_r)\rangle$.

(d) Using the previous part, characterize the minimal generating set of $\text{In}_\preceq(I)$ in terms of a reduced Gröbner basis for I.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use Buchberger's algorithm to obtain a reduced Gröbner basis for

$$I = \langle x^3 - wy^2, x^{10} - w^7z^3 \rangle \subseteq \Bbbk[x, y, z, w]$$

under the grevlex term order.

(H2) Argue that if I is a binomial ideal (i.e., I can be generated by differences of monomials) and $\preceq$ is any term order, then the reduced Gröbner basis of I with respect to $\preceq$ is comprised entirely of binomials.

Then, determine whether the ideal

$$I = \langle x^2 - y^2, x^3y^4 - xy^6 + x^3y, x^4y^3 - x^2y^5 + xy^3 \rangle \subseteq \Bbbk[x, y, z]$$

can be generated by binomials.

(H3) Do (at least) one of the following.

(a) Fix an ideal $I \subseteq \Bbbk[x_1, \ldots, x_k]$ and a term order $\preceq$. Prove that I has a **unique** reduced Gröbner basis under $\preceq$.

(b) Fix an ideal $I \subseteq \Bbbk[x_1, \ldots, x_k]$, let $\preceq$ denote the lex term order with $x_1 \preceq \cdots \preceq x_k$, and let G denote a Gröbner basis for I. Prove that $G' = G \cap \Bbbk[x_2, \ldots, x_k]$ is a Gröbner basis for $I' = I \cap \Bbbk[x_2, \ldots, x_k]$ with respect to $\preceq$ (this is called *elimination*). Does the same hold if $\preceq$ is the glex term order?

(c) Prove that for any homogenous ideal $I \subseteq \Bbbk[x_1, \ldots, x_k]$ under the standard grading and any term order $\preceq$, we have

$$\mathrm{Hilb}(R/I; z) = \mathrm{Hilb}(R/\operatorname{In}_{\prec}(I); z).$$

Does the same necessarily hold if we use a grading other than the standard grading?

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Under any term order $\preceq$ on $\Bbbk[x_1, \ldots, x_k]$, for each variable x_i there are only finitely many monomials x^a such that $x^a \preceq x_i$.

(b) Reverse lexicographic order, defined in the same manner as the grevlex term order but without the initial "total degree" comparison, is a term order.

(c) For $R = \Bbbk[x, y]$, the given generating set for the submodule

$$\big\langle (xy + 4x)e_1 + x^2e_3,\ (y - 1)e_2 + (x - 2)e_3 \big\rangle \subseteq R^3$$

is a Gröbner basis under both the pot-lex term order and the top-lex term order.