Spring 2022, Math 621: Week 11 Problem Set
Due: Thursday, April 21st, 2022
Noetherian Rings and the Hilbert Basis Theorem

Discussion problems. The problems below should be worked on in class.

(D1) *Noetherian rings.*

(a) Play a few rounds of the "ascending chain game": starting with $I_1 = \langle x^2y^3 \rangle \subset \Bbbk[x, y]$, take turns selecting the next ideal in the chain by adding one additional generator (which may or may not override existing generators). Verify that you eventually get "trapped" and become unable to add further elements (i.e., that the ascending chain condition holds). For simplicity, you may restrict your attention to monomial ideals. If you are feeling adventurous, though, give binomnial ideals a try!

(b) In the previous part, what was the longest ascending chain you constructed? Can it be made arbitrarily long? Why does this not contradict the Noetherianity of $\Bbbk[x, y]$?

(c) Prove that if R is Noetherian, then for any infinite list $f_1, f_2, \ldots \in R$ of elements, we have $\langle f_1, f_2, \ldots \rangle = \langle f_1, \ldots, f_N \rangle$ for some N (in particular, any infinite generating set for an ideal I has a finite subset that also generates I).
Note: this is the underlying reason Buchberger's algorithm terminates.

(d) Prove that if R is Noetherian and I is any ideal, then R/I is Noetherian.

(D2) *Modules over Noetherian rings.* In this problem, we will prove the following.

Theorem. *For any ring R, the following are equivalent.*

(a) R is Noetherian;
(b) for every finitely generated R-module M, every submodule $M' \subseteq M$ is also finitely generated; and
(c) for every finitely generated R-module M, any ascending chain

$$M'_1 \subseteq M'_2 \subseteq M'_3 \subseteq \cdots$$

of submodules of M eventually stabilizes.

(a) Demonstrate that $\Bbbk[x_1, x_2, \ldots]$ violates all parts of the above theorem.

(b) Adapt the proof from class to prove that (b) and (c) above are equivalent.

(c) Give a 1-line proof that (b) implies (a) above.

(d) All that remains is to show (a) implies (b). Suppose every ideal in R is finitely generated, and let $M = \langle m_1, \ldots, m_k \rangle$ denote a finitely generated R-module.

(i) Locate a "natural" homomorphism $\varphi : R^k \to M$ from the free module R^k to M, and argue that any submodule $M' \subseteq M$ is finitely generated if its preimage $\varphi^{-1}(M') \subseteq R^k$ is finitely generated. Conclude that it suffices to assume $M = R^k$ is free and each $m_i = e_i$.

(ii) Fix a submodule $M' \subseteq R^k$, and let $I = \{a_1 : (a_1, a_2, \ldots, a_k) \in M'\}$. Prove that I is an ideal of R.

(iii) Prove that $M'' = \{(0, a_2, \ldots, a_k) \in M' : a_2, \ldots, a_k \in R\}$ is a submodule of R^k.

(iv) Argue that M'' is isomorphic to a submodule of the free module R^{k-1}, and that by induction on k, we can assume M'' is finitely generated.

(v) Write $I = \langle b_1, \ldots, b_r \rangle$ for some $b_1, \ldots, b_r \in R$ (why can we do this?), and fix a list of elements $m'_1, \ldots, m'_r \in M'$ with first coordinates $b_1, \ldots, b_r$, respectively. Additionally, write $M'' = \langle m''_1, \ldots, m''_t \rangle$ for some elements $m''_1, \ldots, m''_t \in M$. Prove that $M' = \langle m'_1, \ldots, m'_r, m''_1, \ldots, m''_t \rangle$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) An ideal $I \subseteq R$ is *irreducible* if $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

(a) Prove that any prime ideal is irreducible.

(b) Prove that if R is Noetherian, then every ideal $I \subseteq R$ can be written as an intersection of finitely many irreducible ideals (such an expression is called an *irreducible decomposition* of I).

(H2) Fill in the details in the "proof outline" of the Hilbert Basis Theorem from Tuesday's class. Once you have completed this, it suffices to write "DONE" as your answer.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If R is a ring, $I \subseteq R$ is an ideal, and R/I is Noetherian, then R is Noetherian.

(b) In a Noetherian ring, every ideal can be written as an intersection of finitely many prime ideals.

(c) Any Noetherian ring R also satisfies the descending chain condition (DCC) on ideals (that is, any descending chain

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

of ideals eventually stabilizes).