Spring 2022, Math 621: Week 12 Problem Set
Due: Thursday, April 28th, 2022
Free Resolutions and the Hilbert Syzygy Theorem

Discussion problems. The problems below should be worked on in class.

(D1) *Existence of free resolutions.* Throughout this problem, unless otherwise stated, R is any ring, and modules over R **need not** be finitely generated.

(a) Suppose $R = \Bbbk[x, y]$ and $M = R/\langle x^2 - xy, xy^2 - y^3\rangle$. Find a free resolution for M.

(b) Argue that the sequence $0 \to M \xrightarrow{\varphi} N$ is exact if and only if φ is injective. State and prove analogous results concerning (i) an exact sequence $M \xrightarrow{\varphi} N \to 0$, and (ii) an exact sequence $0 \to M \xrightarrow{\varphi} N \to 0$.

(c) A *short exact sequence* is an exact sequence of the form $0 \to K \to M \xrightarrow{\varphi} N \to 0$. Prove that $K \cong \ker\varphi$ and $N \cong M/\ker\varphi$ in any such sequence.

(d) Given below is a proof that for any ring R, any R-module M has a free resolution. Augment the proof to show that if R is Noetherian and M is a finitely generated R-module, then we can pick the free modules F_i to have finite rank.

Proof. Pick any generating set $G \subset M$ for M (at worst, we could choose $G = M$). Let $F_0 = \bigoplus_{g\in G} R$, and begin the free resolution for M with $0 \leftarrow M \xleftarrow{\varphi_0} F_0$ using the surjective homomorphism $\varphi_0 : F_0 \to M$ defined by sending $\mathbf{e}_g \mapsto g$ for each $g \in G$. Inductively, suppose

$$0 \leftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} \cdots \xleftarrow{\varphi_j} F_j$$

is exact. As before, choose a generating set G for $\ker\varphi_j \subset F_j$, let $F_{j+1} = \bigoplus_{g\in G} R$, and define $\varphi_{j+1} : F_{j+1} \to F_j$ by $\mathbf{e}_g \mapsto g$ for each $g \in G$. This yields exactness at F_j and thus, by induction, the desired free resolution. □

(D2) *Graded free resolutions.* Suppose $R = \Bbbk[x_1, \ldots, x_k]$ is $\mathbb{Z}^d_{\geq 0}$-graded, and fix graded R-modules M and N. We say a map $\varphi : M \to N$ is *graded* if $\deg(f) = \deg(\varphi(f))$ for every homogeneous $f \in M$.

(a) Prove if φ is graded, then $\ker\varphi$ is a homogeneous submodule of M.

(b) Retrace through the proof in Problem (D1)(d). Conclude that every graded module M over a graded ring R has a free resolution in which each F_i and each φ_i is graded (we say such a free resolution is *graded*).

(c) The ideal in Problem (D1)(a) is homogeneous under the standard grading, but technically the free resolution we constructed is **not** graded. Why?

(d) Let's resolve this issue (pun intended). Find a free resolution of $\langle x^2y\rangle \subset \Bbbk[x, y]$. What degree must each monomial in $F_0 = R$ have for this resolution to be graded under the standard grading?

(e) The grading of F_0 in the previous problem is called a *shifted grading*, where we effectively "translate" the grading by some amount. Notationally, we write $R(a)$ to indicate that $\deg(x^b) + a$ is the original degree of $x^b \in R$. Determine the value of a so that

$$0 \longleftarrow \langle x^2y\rangle \longleftarrow R(a) \longleftarrow 0$$

is a (standard) graded free resolution (be careful!).

(f) Returning to the free resolution constructed in Problem (D1)(a), identify the appropriate graded shift of each **summand** of each F_i, and verify that with those choices, the resolutions are graded. Do the same with each free resolution from Tuesday's class.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each of the following rings R and R-modules M, find a (minimal) graded free resolution of M. Be sure to argue as best you can that the image of each map indeed equals the kernel of the next!

(a) $R = \Bbbk[x, y]$ and $M = R/\langle x^3, y^3, x^2y - xy^2\rangle$, under the standard grading.

(b) $R = \Bbbk[x, y, z]/\langle xy, xz, yz\rangle$ and $M = R/\langle x\rangle$, under the fine grading.

(H2) Given a graded R-module M, the *graded Betti number* $\beta_{i,a}(M)$ equals the number of summands of $R(-a)$ appearing in F_i in a **minimal** graded free resolution

$$0 \leftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots$$

for M (recall that in a *minimal* graded free resolution, the matrices defining the maps φ_i have no nonzero constant entries).

Suppose $I \subseteq R = \Bbbk[x, y]$ is a monomial ideal. Obtain a minimal free resolution for R/I, and characterize the (finely) graded Betti numbers in terms of the staircase diagram of I.

(H3) Fix a field $\Bbbk$ and an exact sequence

$$0 \longleftarrow V_0 \longleftarrow V_1 \longleftarrow \cdots \longleftarrow V_\ell \longleftarrow 0$$

of finite dimensional vector spaces over $\Bbbk$. Prove that

$$\sum_{i=0}^{\ell} (-1)^i \dim(V_i) = 0.$$

Hint: as with most good linear algebra results, the rank-nullity theorem is key!