Spring 2022, Math 621: Week 13 Problem Set
Proving Hilbert's Theorem

Discussion problems. The problems below should be worked on in class.

(D1) *The Hilbert Syzygy Theorem.* Let $R = \mathbb{k}[x_1, \ldots, x_k]$. Our goal is to prove the following.

Theorem (Hilbert Syzygy Theorem)**.** *Every finitely generated R-module M has a free resolution of length at most k, one that is graded if M is graded.*

(a) Given an R-module $M = \langle f_1, \ldots, f_r \rangle$, the *syzygy module* is a submodule of R^r given by

$$\mathrm{Syz}(f_1, \ldots, f_r) = \{(a_1, \ldots, a_r) \in R^r : a_1 f_1 + \cdots + a_r f_r = 0\} \subseteq R^r.$$

Verify that $\mathrm{Syz}(f_1, \ldots, f_r) = \ker \varphi$, where $\varphi : R^r \to M$ is given by $\mathbf{e}_i \mapsto f_i$. Note that the syzygy module depends on the choice of **generating set** for M!

(b) Locate the module $\mathrm{Syz}(x^4, x^2y^2, xy^3)$ in the free resolution from lecture.

(c) Fix a Gröbner basis $g_1, \ldots, g_r$ for M with respect to some term order $\preceq$. Fixing i, j, Buchberger's criterion implies we can write

$$S(g_i, g_j) = a_1 g_1 + \cdots + a_r g_r \qquad \text{for some} \qquad a_1, \ldots, a_r \in R^r.$$

Letting $L = \mathrm{lcm}(\mathrm{In}_\preceq(g_i), \mathrm{In}_\preceq(g_j))$, define

$$s_{ij} = \frac{L}{\mathrm{In}_\preceq(g_i)}\mathbf{e}_i - \frac{L}{\mathrm{In}_\preceq(g_j)}\mathbf{e}_j - a_1\mathbf{e}_1 - \cdots - a_r\mathbf{e}_r \in R^r.$$

Verify that $s_{ij} \in \mathrm{Syz}(g_1, \ldots, g_r)$.

(d) Let $J = \langle x^2 - y^2, xy^3 - x^2y, y^5 - x^3y \rangle$. Read Schreyer's Theorem, then use it to find a free resolution for J. Hint: the given generating set for J is a glex Gröbner basis.

Theorem (Schreyer)**.** *Given a Gröbner basis $G = \{g_1, \ldots, g_r\}$ for M under any term order $\preceq$, the elements s_{ij} form a Gröbner basis for $\mathrm{Syz}(g_1, \ldots, g_r)$ under a new term order $\preceq_G$ defined as follows: set $x^a\mathbf{e}_i \preceq_G x^b\mathbf{e}_j$ whenever*

(i) $\mathrm{In}_\preceq(x^a g_i) \preceq \mathrm{In}_\preceq(x^b g_j)$*; or*

(ii) $\mathrm{In}_\preceq(x^a g_i) = \mathrm{In}_\preceq(x^b g_j)$ *and* $i \geq j$.

In particular, $\mathrm{Syz}(g_1, \ldots, g_r) = \langle s_{ij} : 1 \leq i, j \leq r \rangle$.

(e) We will now prove the Hilbert Syzygy Theorem by arguing that the free resolution resulting from Schreyer's Theorem has length at most k.

(i) Suppose $g_1, \ldots, g_r$ is a Gröbner basis for M under some term order $\preceq$, and that $\mathrm{In}_\preceq(g_1), \ldots, \mathrm{In}_\preceq(g_r)$ are in decreasing **top-lex** order. Argue that if for some $m < k$, the variables $x_1, \ldots, x_m$ are absent from each $\mathrm{In}_\preceq(g_i)$, then the variables $x_1, \ldots, x_{m+1}$ are absent from each $\mathrm{In}_{\preceq_G}(s_{ij})$.

(ii) By induction and the previous part, decide what we can conclude for some $\ell < k$ about the initial terms in the resulting Gröbner basis $h_1, \ldots, h_r$ for $\ker \varphi_\ell$ in

$$0 \leftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_\ell} F_\ell.$$

(iii) Prove that $\mathrm{Syz}(h_1, \ldots, h_r) = 0$. Conclude that the Hilbert syzygy theorem holds.

(f) Argue that if $M \subseteq R^n$ is a graded R-module, and $g_1, \ldots, g_r \in M$ form a homogeneous Gröbner basis for M, then each $s_{ij} \in \mathrm{Syz}$ is homogeneous as well. Conclude that Schreyer's Theorem yields a graded free resolution if M is graded.

(D2) *Hilbert's Theorem.* We are finally ready to prove Hilbert's theorem, once and for all.

Theorem (Hilbert)**.** *Fix a finitely generated* $\Bbbk$*-algebra* R*, graded by* $\mathbb{Z}^d_{\geq 0}$*, with homogeneous generators* $y_1, \ldots, y_k \in R$ *of degrees* $r_1, \ldots, r_k$*, respectively, and fix a finitely generated graded* R*-module* M*. The Hilbert series of* M *has the form*

$$\operatorname{Hilb}(M; z) = \frac{h(z)}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}$$

for some polynomial $h(z)$ *with integer coefficients.*

(a) Argue that for some $\mathbb{Z}^d_{\geq 0}$-grading on $T = \Bbbk[x_1, \ldots, x_k]$ and some homogeneous ideal I, we have $R \cong T/I$ as graded rings. Note: you must specify I **and** the grading on T!

(b) Argue that M is a graded T-module.

(c) Argue that M has a graded free resolution of the form

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_k \longleftarrow 0$$

where each F_i is a free T-module.

(d) Use a homework problem from last week to argue that

$$\operatorname{Hilb}(M; z) = \sum_{i=0}^{k} (-1)^k \operatorname{Hilb}(F_i; z).$$

(e) Explain why

$$\operatorname{Hilb}(T; z) = \frac{1}{(1 - z^{r_1}) \cdots (1 - z^{r_k})}.$$

Is it true that $\operatorname{Hilb}(R; z) = \operatorname{Hilb}(T; z)$?

(f) Conclude that Hilbert's theorem holds, and breathe a sign of relief knowing that your life is finally complete.