Spring 2022, Math 621: Project Topics

The goal of each project is to learn about a topic not discussed in class. Throughout the semester, the following will be expected.

- Choose a topic. Please speak with me before making your decision, to ensure it is an appropriate level and so that we can narrow down a reasonable set of goals. You should choose a topic (and have it approved) no later than **Friday, March 25th**.
- Begin reading the agreed-upon background material. Plan to **meet at least twice** with me throughout the rest of the semester, to ensure that you are on track.
- Write (in LaTeX) a paper aimed at introducing your topic to fellow students, containing ample examples and explanations in addition to any theorems and proofs you give. Your writing should convey that you understand the intricacies of any proofs presented. Keep the following deadlines in mind as you proceed.
  - A rough draft of the paper will be due **Monday, May 2nd** (the start of the last week of class). This will be peer reviewed by a fellow student.
  - The final paper will be due on **Thursday, May 12th** (our "final exam" day).
- Optionally, give a 10-15 minute presentation introducing the main ideas of your topic. Presentations will take place during the final exam slot at the semester's end. You should keep in mind your target audience and time constraints when deciding what and how to present. **No extra credit** will be offered for completing a presentation; your reward is the experience presenting mathematics.

  Note: although the presentation is optional, everyone is encouraged to present and should strongly consider doing so.
- Your final grade on the project will be determined by the content, quality, and completeness of your final writeup.

Given below are several project ideas. Many of the listed sources contain more material than is necessary for the project, so be sure to meet with me so we can set reasonable project goals. I am also open to projects not listed here, but you must run them by me before making a decision. Don't be afraid to ask questions at any point during the project!

Algebras and Ideals

(1) *Edge ideals.* Edge ideals are a family of monomial ideals constructed from graphs. Many algebraic properties and constructions, such as free resolutions, can be obtained from combinatorial properties of their associated graph.

Source: *A beginner's guide to cover and edge ideals* (A. Tuyl).

(2) *Binomial edge ideals.* Binomial edge ideals are a family of ideals constructed from graphs. Like their monomial counterparts, many algebraic properties can be obtained from combinatorial properties of their associated graph.

Source: *Binomial edge ideals: a survey* (S. Madani).

(3) *Hierarchical models and algebraic statistics.* Binomial ideals and semigroups arise in sampling problems from algebraic statistics. Each semigroup element corresopnds to a *fiber* of the model (essentially its set of factorizations), and binomial generators correspond to *Markov moves* within each fiber.

Source: *Algebraic algorithms for sampling from conditional distr.* (P. Diaconis, B. Sturmfels).

(4) *General binomial ideals.* In the paper that started it all, Eisenbud and Sturmfels introduce (general) binomial ideals, and use Gröbner bases to prove several foundational results.

Source: *Binomial ideals* (D. Eisenbud, B. Sturmfels).

(5) *Buchberger graphs.* For monomial ideals in 3 variables, a complete free resolution can be constructed from the staircase diagram by embedding a graph into the face of the staircase. This also yields a combinatorial interpretation of all graded Betti numbers.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 3.

(6) *Homology and Betti numbers.* Graded Betti numbers of monomial ideals can be computed using the homology of combinatorially constructed simplicial and polytopal complexes. This transforms the question of constructing a minimal free resolution from an algebraic one into a geometric/topological one.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 4.

Note: this project requires some familiarity with topology and/or homology.

(7) *Alexander duality of monomial ideals.* You may have noticed a visual phenomenon that can occur when staring at 3-variable staircase diagrams in which the staircase appears to turn "inside-out" with outward pointing corners suddently pointing inward and visa versa. This is encapsulated by the concept of *Alexander duality*, a key theorem from algebraic topology.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 5.

Note: this project requires some familiarity with topology and/or homology.

(8) *Gröbner fans.* Gröbner bases (which we will cover later in the semester) are particularly nice generating sets of multivariate polynomial ideals, but they depend on an (often arbitrary) choice of term order. Although there are infinitely many possible term orders, it turns out that, for a fixed ideal I, there are only finitely many (reduced) Gröbner bases. Moreover, if we divide the set of all term orders into equivalence classes based on which Gröbner bases they produce, the result can be interpreted geometrically as a collection of cones, called the *Gröbner fan* of I.

Source: *Computing Groebner Fans* (K. Fukuda, A. Jensen, R. Thomas).

(9) *Cluster algebras.* Cluster algebras are a combinatorially- and geometrically-flavored family of algebras inspired by some surprising recurrences. One classical example is the *pentagon recurrence*: if we let $f_0 = x$, $f_1 = y$, and $f_{n+1} = (f_n + 1)/f_{n-1}$ for $n \geq 1$, then we obtain the (repeating!) sequence

$$x, \quad y, \quad \frac{y+1}{x}, \quad \frac{x+y+1}{xy}, \quad \frac{x+1}{y}, \quad x, \quad y, \quad \ldots$$

Though a fledgling area of research, many of the ideas introduced in our course this semester make appearances in the study of cluster algebras, and often in unexpected ways.

Source: *Root systems and generalized associahedra* (S. Fomin and N. Reading).

Note: this project idea is brought to you by soon-to-be faculty member Gordon Rojas Kirby.

Semigroups and combinatorics

(10) *Hilbert functions and numerical semigroups.* A remarkable number of quantities from the realm of numerical semigroups turn out to exhibit eventually quasipolynomial behavior. Most proofs in the literature are "from first principles" and Hilbert's theorem is only recently starting to be utilized in this context.

Source: *On factorization invariants and Hilbert functions* (C. O'Neill).

(11) *Betti numbers and numerical semigroups.* The Betti numbers of a numerical semigroup algebra can be interpreted in terms of the factorizations of certain elements. Simplicial complexes naturally arise in this setting, as do Hilbert series.

Source: *Betti numbers for numerical semigroup rings* (D. Stamate).

(12) *Semigroups and matroids.* A recent paper considers a question involving both semigroups and matroids. Gröbner bases also make an appearance.

Source: *The monoid of monotone functions on a poset and arithmetic multiplicities for uniform matroids* (W. Bruns, P. García-Sánchez, L. Moci).

(13) *The Kunz polyhdedron.* A numerical semigroup is a subset $S \subseteq \mathbb{Z}_{\geq 0}$ that is closed under addition whose complement in $\mathbb{Z}_{\geq 0}$ is finite. For each $m \geq 3$, there is a polyhedron P_m, shaped like a pointed cone translated slightly from the origin, whose integer points each correspond to a numerical semigroup containing m. The faces of these polyhedra naturally divide the collection of all numerical semigroups containing m (an infinite collection) into finitely many "classes" with similar algebraic structure.

Source: *Wilf's conjecture in fixed multiplicity* (W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilbourne).

Generating functions

(14) *Multivariate quasipolynomials.* We will see that a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is quasipolynomial if it is a polynomial with periodic coefficients (or, equivalently, that its generating function $\sum_{n\geq 0} f(n)z^n$ is rational in z with sufficiently nice numerator and denominator). What about multivariate functions $g : \mathbb{Z}^d_{\geq 0} \to \mathbb{R}$, with d integer inputs instead of just one? What does it mean for g to be a quasipolynomial? The answer to this question turns out to be somewhat more complicated than "a polynomial with periodic coefficients" but has surprising connections to geometry and cones in $\mathbb{Z}^d$.

Source: *Length functions determined by killing powers of several ideals in a local ring* (B. Fields), Chapter 2.

(15) *Combinatorial species.* Our use of generating functions in this class (to concisely express certain quasipolynomial functions) only scratches the surface of their utility. Generating functions are used heavily in combinatorics to give concise answers to counting questions when closed forms are difficult or impossible. The techniques for doing so yield an elegant high-level approach to combinatorics problems and some surprisingly slick proofs. A relatively recent development, "combinatorial species" use ideas and machinery from category theory to provide a general framework for combinatorial proofs.

Source: *Introduction to the Theory of Species of Structures* (F. Bergeron, G. Labelle, P. Leroux), Chapters 1 and 2.

Polytopes

(16) *Matroid polytopes.* Matroids encapsulate the combinatorics of linear independence, and have a knack for arising in unexpected places. Given a matroid M, one can construct a polytope P with one vertex for each basis of M, and the combinatorial properties of P are closely connected to those of M (e.g., the edges of P correspond to basis exchanges, and the facet inequalities of P are obtained from the rank function of M).

Source: *Matroid polytopes and their volumes* (F. Ardila, C. Benedetti, J. Doker).

(17) *The Dehn-Sommerville relations.* The *face numbers* $f_0, f_1, \ldots, f_d$ of a polytope P count the number of faces of P of dimension $0, 1, \ldots, d$, respectively. If P is *simple* (meaning every vertex of P lies in exactly d edges, the smallest number geometrically possible), then the face numbers of P satisfy a collection of equations called the *Dehn-Sommerville relations.* These relations arise frequently in the study of Ehrhart functions of simple polytopes.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 5.

(18) *The Shi arrangement and parking functions.* A collection of hyperplanes $H_1, \ldots, H_k \subseteq \mathbb{R}^d$ can be viewed as cutting space into finitely many "regions" (i.e, the connected components of $\mathbb{R}^d \setminus (H_1 \cup \cdots H_k)$). The Shi arrangement is a particular arrangement of hyperplanes whose regions have especially nice combinatorial structure.

Source: *Parking functions, Shi arrangements, and mixed graphs* (M. Beck, A. Berrizbeitia, M. Dairyko, C. Rodriguez, A. Ruiz, S. Veeneman).