Spring 2023, Math 579: Week 2 Problem Set
Due: Thursday, February 9th, 2023
Pigeon-hole Principle and Inclusion-Exclusion

Discussion problems. The problems below should be worked on in class.

(D1) *Using the pigeon-hole principle.*

Solve each of the following problems using the pigeon-hole principle. Be sure to specify what your boxes and pigeons represent.

(a) Suppose 9 integers are selected at random. Prove that at least 5 have the same parity (even or odd). Is the same true if only 8 integers are selected?

(b) If 10 points are chosen inside of a unit square, then there are two points with a distance at most 0.5 apart.

(c) If 10 points are chosen inside of a unit square, then atleast three points can be covered by a disk of radius 0.5.

(D2) *Applications of Inclusion-Exclusion.* Recall the Sieve formula:

$$|A_1 \cup \cdots \cup A_n| = \sum_{\substack{T \subseteq [n] \\ T \neq \emptyset}} (-1)^{|T|+1} \left| \bigcap_{j \in T} A_j \right|.$$

(a) Write out the Sieve formula **without sigma sums or big intersections** for $n = 3$.

(b) How many positive integers less than 100 are divisible by either 2 or 3?

(c) How many 3-digit positive integers are divisible by 6, 7, or 8? Clearly label A_1, A_2, A_3.

(d) In part (c), for which sets T in the Sieve formula does 24 appear in $\bigcap_{j \in T} A_j$?

(e) How many functions $f : [5] \to [3]$ are surjective? What about surjections $f : [27] \to [4]$?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) How many ways can we arrange the characters $3, 3, 4, 4, 5, 6, 7$ so that no two consecutive digits are identical?

(H2) Find $\phi(210)$, where $\phi(n)$ denote the number of elements of $[n]$ relatively prime to n.

Hint: $210 = 2 \cdot 3 \cdot 5 \cdot 7$.

(H3) Prove that among 1002 positive integers, there are always two integers whose sum or difference is a multiple of 2000.

(H4) Suppose every point in $\mathbb{N}^2$ is colored using one of 8 colors.

(a) Prove that there exists a rectangle whose vertices are monochromatic.

(b) Suppose $\mathbb{N}^2$ is colored using one of r colors, where $r > 0$. For which values of r does part (a) still hold?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose the function $g : \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ satisfies $g(1) = 1$ and

$$\sum_{d \mid n} g(d) = 0$$

for all $n \geq 2$. Find a closed form for $g(n)$ (your answer may use cases, but **not** sums).