Spring 2023, Math 579: Week 3 Problem Set
Due: Thursday, February 16th, 2023
Binomial Theorem and Combinatorial Proofs

Discussion problems. The problems below should be worked on in class.

(D1) *Combinatorial proofs.*

(a) Compare your answers to the preliminary problems, and come to a concensus.

(b) Fill in the blanks in the following **combinatorial** proof that for any $n \geq 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Proof. We will count the number of ways to choose subsets $A, B \subseteq [n]$ with $A \subseteq B$.

First, for each $i \in [n]$, either $i \in A$, $i \in B \setminus A$, or ____. This yields 3^n possibilities.

Second, if we let $k = |B|$, then for each $k = 0, 1, \ldots, n$, there are ____ choices for B, and ____ ways to choose a subset $A \subseteq B$. This yields

$$2^0\binom{n}{0} + 2^1\binom{n}{1} + \cdots + 2^n\binom{n}{n} = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

possibilities. We conclude the claimed identity must hold. □

(c) In the proof of the previous part, replace "$A, B \subseteq [n]$ with $A \subseteq B$" in the first line with "$C, D \subseteq [n]$ with $C \cap D = \emptyset$", and then rewrite the rest of the proof accordingly.

(D2) *More combinatorial proofs.*

(a) Give a combinatorial proof that whenever $0 \leq k \leq n - 3$,

$$\binom{n}{3}\binom{n-3}{k} = \binom{n}{k}\binom{n-k}{3}.$$

(b) Give a combinatorial proof that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Hint: invent a question involving a club with $n + 1$ members (or something strictly more creative, if you like).

Hint: for the right hand side, imagine that **you** are one of the club members.

(c) Give a combinatorial proof that for all $n \geq 1$,

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1.$$

(d) Give a combinatorial proof that for all $n \geq 1$,

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Use induction on n to prove that for all $n \geq 1$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Hint: use the identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

in your inductive step.

(H2) Prove the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

(a) using the formula for binomial coefficients (involving factorials), and

(b) using a combinatorial proof.

(H3) Give a combinatorial proof that for all $n \geq 2$,

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = 2^{n-2}n(n-1).$$

(H4) Give a combinatorial proof that for all $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give a combinatorial proof that for all $n \geq 1$ and all $r \in [n]$,

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$