Spring 2023, Math 579: Week 4 Problem Set
Due: Thursday, February 23rd, 2023
The Binomial Theorem

Discussion problems. The problems below should be worked on in class.

(D1) *The binomial theorem.* Recall the binomial theorem from Tuesday:

$$(x+z)^n = \sum_{k=0}^{n} \binom{n}{k} x^k z^{n-k}.$$

(a) Use the binomial theorem to find the coefficient of $x^9 z^{15}$ in the expression $x^5(x^2 - z)^{17}$.

(b) Use the binomial theorem to prove that for any $n \geq 0$,

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

We call this an **algebraic** proof.

(c) Apply the binomial theorem (3 times!) to the expression

$$(x+1)^{n+1} = x(x+1)^n + (x+1)^n.$$

Then, reindex each sum to contain x^{k+1} (as opposed to x^k) and pull out terms so that each sum starts at $k = 1$ and ends at $k = n$. Lastly, consolidate the right hand side into a single sum. Comparing coefficients on the left and right hand sides, what identity is obtained?

(D2) *The binomial theorem (deep cuts).* The goal of this problem is to prove for all $n \geq 1$,

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} = \frac{1}{n+1}.$$

(a) We can substitute into the binomial theorem to obtain

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Take the antiderivative of both sides of this equality with respect to x.
Note: don't make your Calculus teacher sad, be sure your answer has a $+C$ in it!

(b) The value of C in part (a) must be chosen so that equality holds for **all** values of x. Can we plug in a carefully chosen value of x to determine the value of C?

(c) Use part (b) to give an algebraic proof of the identity at the start of this problem.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the coefficient of $x^{11}z^7$ in the expansion of $(x+z)^{18} + x^3(x-z)^{15}$.

Hint: **don't expand**!!! This is what the binomial theorem is for!!!

(H2) Consider the following identity, valid for all $n \geq 1$:

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = 2^{n-2}n(n-1).$$

Give an **algebraic** proof of this identity (that is, using the binomial theorem).

(H3) Give an **algebraic** proof that for $n \geq 1$,

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}.$$

Hint: write 3 and -1 each in a clever way, and then use the binomial theorem twice.

(H4) Prove that for all $n, k \in \mathbb{Z}_{\geq 1}$ with $k \leq n$,

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} = (-1)^k \binom{n-1}{k}.$$

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Give an **algebraic** proof that for any $n, m \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, we have

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{m+j}{m} = \begin{cases} (-1)^n & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}$$