Spring 2023, Math 579: Week 5 Problem Set
Due: Thursday, March 2nd, 2023
Integer Partitions

Discussion problems. The problems below should be worked on in class.

(D1) *Compositions vs. integer partitions.*

(a) Compare your answers to the prelim problems.

(b) Find the number of (strong) compositions of 10 into all even parts. Compare to (P1).

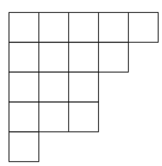
(c) Based on your observations in the previous part, state and prove a conjecture about the number of (strong) compositions of n into even parts.

(d) Show $p(6) = 11$ and $p_3(7) = 4$ by listing the corresponding partitions.

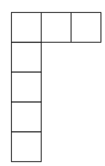
(e) Give a combinatorial proof that for $n \geq 2$, the number of partitions of n with no 1's is given by $p(n) - p(n-1)$.

(D2) *Ferres diagrams.* Recall that $p(n)$ counts the number of integer partitions of n, and $p_k(n)$ counts the number of partitions into exactly k parts.

(a) Given a partition of n, the associated *Ferrer diagram* or *Young diagram* is a drawing of n boxes that are "upper-left justified" where the number of boxes in the i'th row equals the size of the i'th part. See below for examples.



$5 + 4 + 3 + 3 + 1$



$3 + 1 + 1 + 1 + 1$

Find all partitions of 7 into 3 parts, and write the Ferrer diagram for each partition.

(b) Given a Ferrer diagram of a partition, what is the number of rows equal to? What is the number of columns equal to?

(c) Using the previous part, find a bijection between the partitions of n into at most k parts and the partitions of n into parts of size at most k.

(d) Based on your bijection in the previous part, is it true that $p_k(n)$ equals the number of partitions of n into parts of size exactly k?

(e) The partition obtained by reflecting a Ferres diagram about its diagonal (which you likely used in the preceeding few problems) is called the *conjugate* partition. Explain why the conjugate of a partition in which every part appears an even number of times is a partition whose parts are all even. Is the converse true?

(f) Describe the partitions whose conjugate partition has no parts of size 1.

(g) Prove that the number of partitions of n into at most k parts equals $p_k(n + k)$.

(h) A partition is *self-conjugate* if it equals its conjugate. Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that for $n \geq 2$, the number of partitions of n in which the two largest parts are equal is given by $p(n) - p(n-1)$.

(H2) A partition is *self-conjugate* if it equals its conjugate. Prove the number of self-conjugate partitions of n has the same parity as $p(n)$ (that is, the number of self-conjugate partitions of n is even if and only if $p(n)$ is even).

(H3) Find an expression for the number of partitions of n in which each part appears an even number of times (your expression is allowed to use $p(k)$ for $k \leq n$). For example, this includes the partition $4+4+3+3+3+3+1+1$, but not $8+3+3+1+1$.

(H4) Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

(H5) Let a_n denote the number of strong compositions of n into parts that are larger than 1. For $n \geq 4$, find a formula for a_n in terms of a_{n-1} and a_{n-2}.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.