

Spring 2023, Math 579: Week 6 Problem Set
Due: Thursday, March 9th, 2023
Set Partitions

Discussion problems. The problems below should be worked on in class.

(D1) *A Pascal-like recurrence.* Recall that $S(n, k)$ denotes the number of set partitions of $[n]$ into exactly k blocks (called the *Stirling numbers of the 2nd kind*).

- (a) Find $S(5, 4)$, $S(4, 3)$, and $S(4, 4)$ by listing set partitions.
- (b) Verify $S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$ holds for $n = 5$, $k = 4$.
- (c) Give a combinatorial proof that, for $n - 1 \geq k \geq 1$, we have

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k).$$

Hint: count the ways to partition an n -person class *you are in* into k nonempty groups.

(D2) *Bell numbers.* Let $B(n)$ denote the total number of set partitions of $[n]$, that is,

$$B(n) = \sum_{k=1}^n S(n, k)$$

(called the n -th *Bell number*).

- (a) Find $B(1)$, $B(2)$, $B(3)$, and $B(4)$. Which of these did you find in the prelim problem?
- (b) Find $B(5)$ using the identity

$$B(n + 1) = 1 + \sum_{k=1}^n \binom{n}{k} B(k).$$

- (c) Give a combinatorial proof of the identity in part (b).
- (d) Give a combinatorial proof that for all $n \geq 1$,

$$B(n + 1) - B(n) = \sum_{k=1}^n kS(n, k).$$

(D3) *An unfortunate formula for Stirling numbers.* For $n \geq k \geq 1$, let $O_{n,k}$ denote the total number of surjective functions $[n] \rightarrow [k]$ (that is, functions in which every element of $[k]$ is the image of some element of $[n]$).

- (a) Find **all** functions $[3] \rightarrow [2]$. Label each as “surjective” or “not surjective” accordingly.
- (b) Find a formula for the total number of functions $[n] \rightarrow [k]$.
- (c) Show that $O_{5,2} = 5^2 - 2$. Hint: why is it expressed like this?
- (d) Argue that $O_{4,3} = 4^3 - \binom{3}{2}4^2 + \binom{3}{1}4^1$.
- (e) Using the idea in the previous part, find an expression for $O_{5,3}$.
- (f) For $n \geq k$, use the Sieve formula to show that

$$O_{n,k} = \sum_{j=0}^k (-1)^j \binom{k}{j} (k - j)^n.$$

Hint: first, verify this expression matches what you found in the previous part.

- (g) Verify $O_{4,3} = \binom{4}{2} \cdot 3 \cdot 2$. Hint: why is it expressed like this?
- (h) Give a combinatorial proof that $S(n, k) = O_{n,k}/k!$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

- (H1) Use the recurrence identity $S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k)$ from class to complete the next 3 rows of the triangle of Stirling numbers of the 2nd kind (the analog of Pascal's triangle where the entry in row n and position k is $S(n, k)$).

$$\begin{array}{ccccccc}
 & & & & 1 & & & & \\
 & & & & 1 & & 1 & & \\
 & & & 1 & & 3 & & 1 & \\
 & & 1 & & 7 & & 6 & & 1
 \end{array}$$

- (H2) Find a closed formula for $S(n, n - 2)$ in terms of n , valid for $n \geq 3$.
- (H3) Find a formula for the number of partitions of $[n]$ into blocks of size exactly 2.
- (H4) Let $F(n)$ denote the number of set partitions of $[n]$ with no singleton blocks. Prove that $B(n) = F(n) + F(n + 1)$.
- (H5) Find a recursive formula for $F(n + 1)$ in terms of $F(k)$ for $k \leq n$, and give a combinatorial proof of its correctness.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

- (C1) Give a combinatorial proof that

$$B(n + m) = \sum_{j=1}^m S(m, j) \left(j^n + \sum_{k=1}^n \binom{n}{k} j^{n-k} B(k) \right).$$

- (C2) Give a combinatorial proof that

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k - j)^n.$$

Hint: Problem (D3) provides a potential path to complete this problem.