Spring 2023, Math 579: Week 11 Problem Set
Due: Thursday, April 20th, 2023
Trees

Discussion problems. The problems below should be worked on in class.

(D1) *Counting walks of fixed length.* Let G be a graph G and let A be its adjacency matrix.

(a) Compare your answers to the prelim problems.

(b) Find the adjacency matrix of K_5, the complete graph on 5 vertices. Verify that the entry $(A^2)_{3,4}$ equals the number of walks from vertex 3 to vertex 4 of length 2.

(c) Recall that the determinent of an upper-triangular matrix is the product of the diagonal entries. Find the determinent of the matrix in Preliminary Problem (P3) by first adding a multiple of one row to another and then using this fact.

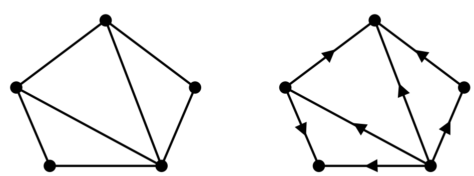
(d) Use the Matrix-Tree Theorem from class to find the number of spanning trees of K_5. Hint: for the determinent step, start by adding every row to the first row (which doesn't change the determinent).

(D2) *Counting spanning trees.* Fix a directed graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. The *incidence matrix* of G is the $n \times m$ matrix M defined by

$$M_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j; \\ -1 & \text{if } v_i \text{ is the tail of } e_j; \\ 0 & \text{otherwise} \end{cases}$$

Note that this is **different from the adjacency matrix** of G.

(a) Find all spanning trees in the undirected graph depicted on the left below.



(b) Find the incidence matrix M of the directed graph depicted on the right above.

(c) Consider the matrix M_0 obtained by omitting the last row of M. Compute the determinant of several 4×4 submatrices of M_0 (divide the work on this!).

(d) Notice that the value of each determinant in part (c) is either 0 or ± 1. Using the edges corresponding to the columns, formulate a conjecture as to when this value is nonzero.

(e) Fix an arbitrary directed graph G with incidence matrix M, and let M_0 denote the matrix obtained by removing the last row of M. The Binet-Cauchy formula tells us

$$\det(M_0 M_0^T) = \textstyle\sum_B (\det B)^2;$$

where the sum ranges over all $(n-1) \times (n-1)$ submatrices B of M_0. Use this and part (d) to show $\det(M_0 M_0^T)$ equals the number of (undirected) spanning trees of G.

(f) Compute the matrices MM^T and $M_0 M_0^T$ for the graph in part (a). Do these matrices look familiar? Use this to prove the Matrix Tree Theorem.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find all non-isomorphic trees on 6 vertices.

(H2) How many different trees are there on $[n]$ whose vertices have degree at most 2? How many such trees are there up to isomorphism?

(H3) Prove that in any tree G, any two longest paths cross each other. Is the same true if G is connected but not necessarily a tree?

(H4) Suppose G is a tree, and no vertex of G has degree more than 3. Prove that the number of vertices with degree 1 is two more than the number of vertices with degree 3.

(H5) Find the number of spanning trees of the circle graph C_n. Verify your answer using the matrix tree theorem.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Find the number of spanning trees of the wheel graph W_n.